EUCLID.

AN EPITOME

OF THE

FIRST THREE BOOKS

OF

EUCLID'S

ELEMENTS OF GEOMETRY.

H. M. S. BRITANNIA.

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THIS epitome of the first three books of Euclid is prepared with the design of presenting the propositions to the Cadets of H.M.S. "Britannia" in a form of reasoning easily followed, and of establishing an uniformity in the manner of writing them out.

Many "Problems" as they can manifestly be performed with instruments, and some "Theorems" which are *converse* to others previously demonstrated, are omitted in order to reduce the subject to a limit consistent with the time which can be devoted to it.

EUCLID.

DEFINITIONS.

A **Point** has no parts or magnitude, it indicates *position* only.

A **Line**  has length only without breadth.

The extremities of a line are points.

A straight line ——— lies evenly between its extremities.

A **Superficies** (*surface*) has length and breadth only, *without thickness or depth.*

A **Plane Superficies** (commonly called a "Plane") is that in which any two points being taken the straight line between them lies wholly in that superficies—*as the surface of a slate.*

A **Figure** is that which is enclosed by lines or surfaces.

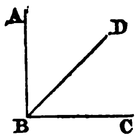
A **Plane Figure** is wholly contained in a plane—*as a circle or a square.*

A **Solid Figure** has length, breadth, and thickness.

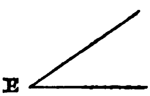
Rectilineal Figures are contained by straight lines.

Equilateral Figures have all their sides equal.

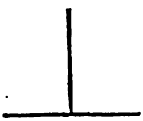
A **Plane Rectilineal Angle** is the inclination of two straight lines to one another which meet together, but are not in the same straight line.



There being more than one angle at a point as at B, where there are three angles, any one of them is expressed by placing B between the letters at the other extremities of the two straight lines which contain the angle. Thus, *the whole angle* at B is expressed by ABC or CBA, the upper one of the two remaining angles by ABD or DBA: the lower one by CBD or DBC.



When there is only one angle at a point, it may be named by the single letter at that point, as "angle E."

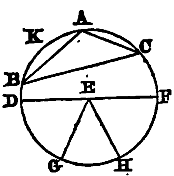


When a straight line meets another straight line, making the adjacent angles equal to each other, each of the angles is called a **right angle,** and the straight lines are said to be **at right angles** or **perpendicular to** each other.

A Straight Line is perpendicular to a Plane when it makes right angles with every straight line meeting it in that plane.

An **Obtuse Angle** is greater than a right angle.

An **Acute Angle** is less than a right angle.



A **Circle** is a plane figure contained by one line, which is called the circumference, and is such that all the **radii** ED, EG, EH, or straight lines drawn from a certain point E (called the **centre**) to the circumference, are equal to one another.

A **Diameter** of a circle is a straight line, as DF, passing through the centre, and terminated both ways by the circumference.

A **Semicircle** (*half circle*) is the figure contained by a diameter and the part of the circumference it cuts off, as the figure DAFED.

Equal Circles are those whose diameters are equal or whose radii are equal.

Any straight line in a circle which is terminated both ways by the circumference, as the straight line BC, is called a **Chord.**

A part of the circumference of a circle, as CF, is called an **Arc.**

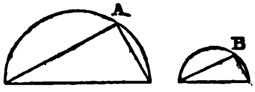
A **Segment** of a circle is the figure contained by a chord and the arc it cuts off, as the figure BKCB.

An **Angle in a Segment** is the angle contained between two straight lines drawn from any point in the arc of the segment to the extremities of the chord, as the angle BAC.

An angle is said to *stand upon* the arc intercepted between the straight lines which contain the angle, as the angle BAC *stands upon* the arc BGFC.

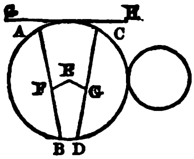
A **Sector** of a circle is the figure contained by two radii and the arc between them, as the figure EGHE.

Similar Segments of circles are those in which the angles are equal,



(as when angle A equals angle B.)

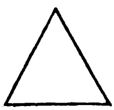
A Straight Line, as GH, is said to **touch** a circle when it meets it, and, being producèd, does not cut it.



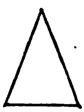
Circles are said to **touch** one another when they meet, but do not cut one another.

Straight Lines (as AB, CD) are **equally distant** from the centre of a circle when the perpendiculars, EF, EG, drawn to them from the centre are equal; and a straight line on which a greater perpendicular falls is farther from the centre.

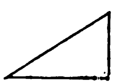
A rectilineal figure is said to be **inscribed in a circle**, when all its angular points are situated in the circumference of the circle.



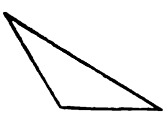
A **Plane Triangle** is a figure contained by three straight lines, *and has consequently three plane rectilineal angles.*



An **Isosceles Triangle** has two equal sides.

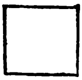


A **Right-angled Triangle** has one of its angles a right angle.

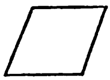


An **Obtuse-angled Triangle** has one of its angles an obtuse angle.

Parallel Straight Lines are such as are in the same plane, and which, being produced ever so far both ways, never meet.



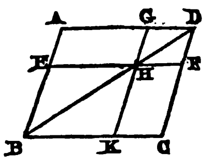
A **Square** is a quadrilateral (*four-sided*) figure, whose sides are all equal and angles all right angles.



A **Parallelogram** is a quadrilateral figure whose opposite sides are parallel.

A **Rectangle** is a right-angled parallelogram, and is said to be contained by any two of the straight lines, which contain one of the right angles.

A quadrilateral figure of no regular shape is called a **trapezium.**

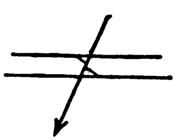


A straight line BD joining two opposite angles of a quadrilateral figure is called a "**diagonal.**"

A quadrilateral figure may be named by the two letters at opposite angular points, as AC, GF.

If ABCD be a parallelogram, in which GK is parallel to AB or DC and EF parallel to AD or BC, then the figures EK, GF, are *parallelograms about the diagonal*, and the remaining figures AH, HC, are the "complements."

N.B.—*Removing from the whole figure one of the parallelograms about the diagonal, we have a gnomon.*



When a straight line cuts two other straight lines, two angles, interior as to the two straight lines, and on opposite sides of the cutting line, are called **alternate angles.**

Theorem, a geometrical proposition to be proved by reasoning.

Problem, a geometrical operation to be performed.

Corollary, a consequence which follows from a preceding proposition.

Hypothesis, that which is assumed as true for the purpose of argument.

POSTULATES,

Or Problems which may evidently be done.

Any two points may be joined by a straight line.

N.B.—When told to **join** two points, it is understood we are to join them by a straight line.

A terminated straight line may be produced to any length in a straight line.

A circle may be described from any centre at any distance from the centre.

A straight line may be drawn from any point in any direction equal to a given straight line. (I. 2.)

From the greater of two given straight lines a part may be cut off equal to the less. (I. 3.)

AXIOMS,

Or truths which the mind immediately accepts, and which, therefore, require no demonstration.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Things which are double of the same, are equal to one another.

7. Things which are halves of the same, are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot inclose a space.

11. All right angles are equal to one another.

12. If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

13. The squares described on *equal* straight lines are themselves equal.

ABBREVIATIONS.

à fortiori	*much more*
adj^{t}	*adjacent*
alt.	*alternate*
comp^{t}	*complement*
conseq.	*consequently*
constr.	*construction*
cont^{d}	*contained*
Cor.	*corollary*
desc.	*describe*
def.	*definition*
div^{d}	*divided*
diag^{l}	*diagonal*
diam^{r}	*diameter*
ea. to ea.	*each to each*
ext^{r}	*exterior*
ext^{y}	*extremity*
fig.	*figure*
hyp.	*hypothesis*
i. e.	*that is*
insc^{d}	*inscribed*
int^{r}	*interior*
opp.	*opposite*
prod.	*produce*
prod^{d}	*produced*
prop.	*proposition*
p^{t}	*point*
quad^{l}	*quadrilateral*
Q. E. D.	"Quod erat demonstrandum" *which was to be demonstrated*
Q. E. F.	"Quod erat faciendum" *which was to be done*
rect.	*rectangle*
rect. AB·BC	*rectangle contained by* AB *and* BC.
rect^{l}	*rectilineal*
rem^{g}	*remaining*
rem^{r}	*remainder*
req^{d}	*required*
resp^{ly}	*respectively*
r^{t} $\angle^{s}$	*right angles*
seg^{t}	*segment*
sim^{ly}	*similarly*
st. line	*straight line*
sq.	*square*
term^{d}	*terminated*
tog.	*together*
viz.	*namely*
wh.	*which*
&c.	&c.

SYMBOLS.

$\angle$	angle.
$\triangle$	triangle.
$\odot$	circle.
$\odot^{ce}$	circumference.
$\square^{m}$	parallelogram.
$=$	equal to.
$>$	greater than.
$<$	less than.
$\perp$	perpendicular to.
$\parallel$	parallel to.
$\parallel^{ls}$	parallel straight lines.
(I. 4)	Book I. Prop. IV.

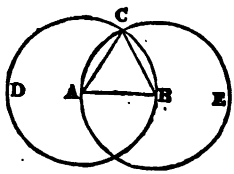
BOOK I.

PROPOSITIONS.

PROP. I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be a given finite st. line: it is req[d] to describe an equilateral △ upon it.



From centre A, at dist. AB, describe ⊙ BCD,
from centre B, at dist. BA, describe ⊙ ACE.
Join AC, BC:
ABC is an equilateral △.
For AC = AB, *being radii of* ⊙ *BCD*,
and BC = AB, *being radii of* ⊙ *ACE*.

But things, which are equal to the same thing, are equal to each other.

∴ AC = BC;
and they are both = AB.

Wherefore △ ABC is equilateral, and it is desc[d] upon the given st. line AB.

Q. E. F.

PROP. II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

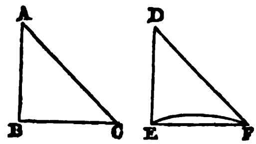
PROP. III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

PROP. IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to each other; they shall likewise have their bases or third sides *equal, and the two triangles shall be equal, and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite.*

Let two Δ^s ABC, DEF have the two sides AB, AC of the one equal to the two sides DE, DF of the other, ea. to ea., viz., AB = DE, AC = DF, and let included $\angle$ BAC = included $\angle$ EDF.



Then, base BC shall = base EF,

△ ABC shall = △ DEF,

and the rem^g ∠^s shall be equal to the rem^g ∠^s, ea. to ea., viz., ∠ ABC = DEF and ∠ ACB = DFE.

For, if △ ABC be applied to △ DEF, so that p^t A may coincide with D, and st. line AB may fall upon DE,

then p^t B will coincide with E, for AB = DE, (*hyp.*)

and, AB coinciding with DE,

AC will fall upon DF, for ∠ BAC = EDF, (*hyp.*)

and p^t C will coincide with F, for AC = DF. (*hyp.*)

Now, B coinciding with E and C with F,

st. line BC must coincide with EF,

otherwise two st. lines would enclose a space,

which is impossible (*axiom* 10).

∴ BC coincides with EF

and △ ABC wholly coincides with △ DEF.

Consequently base BC = base EF,

△ ABC = △ DEF,

∠ ABC = DEF,

∠ ACB = DFE,

or the triangles *are equal in all respects.*

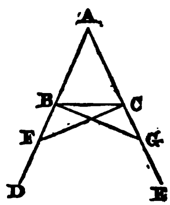
Wherefore, if two triangles, &c.

Q. E. D.

PROP. V. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.

In △ ABC let side AB = side AC
and let AB, AC be prod[d] to D and E.
Then ∠ ABC shall = ACB
and ∠ CBD shall = BCE.



In BD take any p[t] F, and from AE cut off AG = AF. (I. 3)
Join BG, CF.
Now, AB being = AC (*hyp.*), and AG = AF (*const.*),
we have, in the two △[s] ABG, ACF,
two sides AB, AG = two sides AC, AF, ea. to ea.,
and ∠ A common to both △[s].
∴ △[s] ABG, ACF are equal in all respects. (I. 4)
Consequently, base BG = base CF,
∠ ABG = ACF,
and ∠ AGB = AFC.

Again, ∵ AF = AG and part AB = part AC

∴ remainder BF = remainder CG. (*axiom* 3)

Hence, in △ˢ CBF, BCG, we have

two sides, BF, FC = CG, GB, ea. to ea.,

and ∠ BFC = CGB. (*from above*)

∴ △ˢ CBF, BCG are equal in all respects. (I. 4)

Consequently, ∠ CBF = BCG and ∠ BCF = CBG.

Now, *from above*, ∠ ABG = ACF

and ∠ˢ CBG, BCF, parts of them, being equal

∴ remˢ ∠ ABC = ACB (∠ˢ *at the base*),

and it has just been shewn that

∠ CBF = BCG (∠ˢ *on other side of base*).

Wherefore, the angles at the base of an isosceles triangle are equal, &c.

Q. E. D.

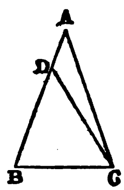
COR. Hence, every equilateral triangle is also equiangular.

PROP. VI. THEOREM.

If two angles of a triangle be equal to one another, the sides also which subtend the equal angles shall be equal to one another.

In △ ABC, let ∠ ABC = ACB.

Then side AB shall equal side AC.



For, if not, suppose AB > AC;

from BA cut off BD = AC, (I. 3); and join DC.

Now, in the △s DBC, ACB,

∵ DB = AC (*const.*) and BC is common to both △s,

we have two sides DB, BC = AC, CB, ea. to ea.,

and ∠ DBC = ACB, (*hyp.*)

∴ △ DBC = △ ACB, (I. 4)

wh. is impossible (*axiom* 9).

∴ AB is not unequal to AC,

i.e. AB = AC.

Wherefore, if two angles of a triangle, &c.

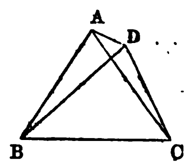
Q. E. D.

Cor. Hence every equiangular triangle is also equilateral.

PROP. VII. THEOREM.

Upon the same base, and on the same side of it, there cannot be two triangles which have the sides terminated in one extremity of the base equal to one another and likewise the sides terminated in the other extremity.

For, supposing it possible, let Δ^s ABC, DBC, upon same base BC and upon same side of it, have the sides, term[d] in one ext[y] of the base, equal to one another, viz., AB = DB; and likewise the sides term[d] in the other ext[y], viz., AC = DC: and, first, let the vertex of ea. Δ be without the other Δ.



Join AD.

Now $\because$ BA = BD, (*hyp.*)

$\therefore$ $\angle$ BAD = BDA; (I. 5)

But $\angle$ BAD > CAD,

$\therefore$ $\angle$ BDA > CAD,

à fortiori $\therefore$ $\angle$ CDA > CAD:

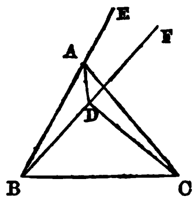
but, again $\because$ CA = CD (*hyp.*),

$\therefore$ $\angle$ CDA = CAD, (I. 5)

$\therefore$ $\angle^s$ CAD, CDA, are both equal and unequal,

wh. is absurd.

Secondly, let one of the vertices as D be within the other △ ABC.



Prod. BA, BD to E, F and join AD.

Now ∵ BA = BD, (*hyp.*)

∴ ∠ EAD = FDA; (I. 5)

but ∠ EAD > CAD,

∴ ∠ FDA > CAD,

à fortiori ∴ ∠ CDA > CAD:

But, again ∵ CA = CD, (*hyp.*)

∴ ∠ CAD = CDA, (I. 5)

∴ ∠ˢ CAD, CDA, are both equal and unequal, wh. is absurd.

Thirdly. The case in wh. the vertex of one △ is upon a side of the other needs no demonstration.

Wherefore upon the same base, &c.

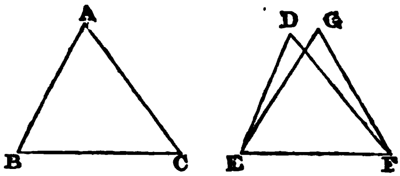
Q. E. D.

PROP. VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle, which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.

Let ABC, DEF be two $\triangle^s$, having the two sides AB, AC = the two sides DE, DF ea. to ea., viz., AB = DE, and AC = DF, and also the base BC = base EF.

Then $\angle$ BAC shall = EDF.



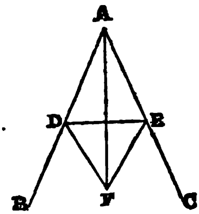
For if $\triangle$ ABC be applied to $\triangle$ DEF,
so that p^t B be on E, and st. line BC on EF;
then p^t C will coincide with F, for BC = EF. (*hyp.*)
Now BC coinciding with EF,
BA, AC must coincide with ED, DF;
for if they do not, but have a different situation as EG, GF, then upon the same base, and upon the same side of it there will be two $\triangle^s$, having their sides, which are terminated in one ext^y of the base, equal to one another, and likewise those sides, which are terminated in the other ext^y;
but this is impossible, (I. 7)
$\therefore$ sides BA, AC coincide with sides ED, DF;
and $\angle$ BAC coincides with $\angle$ EDF
$\therefore$ $\angle$ BAC = EDF. (*axiom* 8)
Wherefore, if two triangles have two sides, &c.

Q. E. D.

PROP. IX. PROBLEM.

To bisect a given rectilineal angle, that is, to divide into two equal angles.

Let BAC be the given rect[l] ∠:
It is req[d] to bisect it.



In AB take any p[t] D;
from AC cut off AE = AD (I. 3) and join DE;
on the side of DE remote from A
describe the equilateral △ DEF, (I. 1), and join AF:
Then AF bisects ∠ BAC.
For in the two △[s] DAF, EAF,
∵ DA = EA (*constr.*) and AF is common to both △[s],
we have two sides DA, AF = two sides EA, AF, ea. to ea.,
and base DF = base EF (*constr.*)
∴ ∠ DAF = EAF: (I. 8)
Wherefore ∠ BAC is bisected by st. line AF.

Q. E. F.

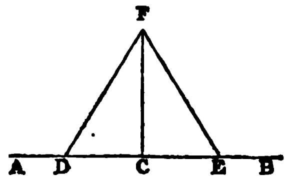
PROP. X. PROBLEM.

To bisect a given finite straight line, that is, to divide it into equal parts.

PROP. XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given st. line, and C a given p^t in it:
It is req^d to draw a st. line from C $\perp$ AB.



In AC take any p^t D, and from CB cut off CE = CD; (I. 3) upon DE describe an equilateral $\triangle$ DEF, (I. 1); and join CF:

Then CF, drawn from C, is $\perp$ AB.

For, in the two $\triangle^s$ DCF, ECF,

$\because$ DC = CE (*constr.*) and CF is common to both $\triangle^s$,

we have two sides DC, CF = two sides EC, CF, ea. to ea.,

and base DF = base EF, (*constr.*)

$\therefore$ $\angle$ DCF = ECF: (I. 8)

and these are adj^t $\angle^s$:

But when the two adj^t $\angle^s$, which one st. line makes with another st. line, are equal to one another, each of them is a r^t $\angle$: (*def.*)

$\therefore$ Each of the $\angle^s$ DCF, ECF is a r^t $\angle$:

Wherefore from the given p^t C, in the given st. line AB, CF has been drawn $\perp$ AB.

Q. E. F.

PROP XII. PROBLEM.

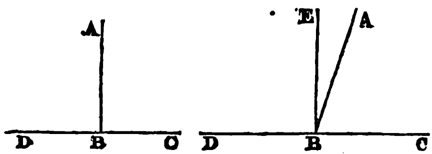
To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

PROP. XIII. THEOREM.

The angles, which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the st. line AB make with CD, upon one side of it, the ∠s ABC, ABD.

These shall be either two right angles
or together equal to two right angles.



For if ∠ ABC = ABD,
Each of them is a rt ∠. (*def.*)
But if ∠ ABC is not = ABD,
from pt B draw BE ⊥ CD. (I. 11)
Now ∠ CBE being = the ∠s CBA, ABE,
add ∠ EBD to each of these equals,
then ∠s CBE, EBD = the three ∠s CBA, ABE, EBD. (*axiom* 2)
Again ∠ DBA being = the ∠s DBE, EBA,
add ∠ ABC to each of these equals,
then ∠s DBA, ABC = the three ∠s DBE, EBA, ABC: (*axiom* 2)
But, *from above*, ∠s CBE, EBD = same three ∠s,
and things, wh. are equal to the same thing, are equal to one another,
∴ ∠s CBE, EBD = the ∠s DBA, ABC,
but ∠s CBE, EBD are two rt ∠s, (*constr.*)
∴ ∠s DBA, ABC are together = two rt ∠s. (*axiom* 1)
Wherefore, when a straight line, &c.

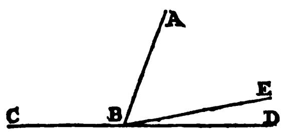
Q. E. D.

PROP. XIV. THEOREM.

If at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At p^t B in st. line AB, let the two st. lines BC, BD, upon opp. sides of AB, make the adj^t $\angle^s$ ABC, ABD together = two r^t $\angle^s$;

Then BD shall be in the same st. line with BC.



For, if BD be not in the same st. line with BC,
let BE be in the same st. line with it.
Then, AB meeting the st. line CBE,
the adj^t $\angle^s$ ABC, ABE are tog. = two r^t $\angle^s$; (I. 13)
but $\angle^s$ ABC, ABD are tog. = two r^t $\angle^s$, (*hyp.*)
$\therefore$ $\angle^s$ ABC, ABE are tog. = ABC, ABD, (*axiom* 1)
take away the common $\angle$ ABC,
$\therefore$ rem^g $\angle$ ABE = rem^g $\angle$ ABD, (*axiom* 3)
which is impossible. (*axiom* 9)
$\therefore$ BE is not in same st. line with BC.

And in same manner it may be shown that no other can be in same st. line with BC, but BD,

$\therefore$ BD is in same st. line with BC.

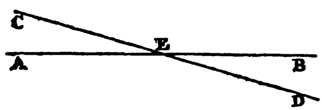
Wherefore, if at a point, &c.

Q. E. D.

PROP. XV. THEOREM.

If two straight lines cut one another, the vertical, or opposite, *angles shall be equal.*

Let the st. lines AB, CD cut one another in p^t E.
Then ∠ CEB shall = AED, and ∠ AEC shall = BED.



For st. line AE meeting st. line CD in p^t E,
the adj^t $\angle^s$ AEC, AED are tog. = two r^t $\angle^s$; (I. 13)
and st. line CE meeting st. line AB in p^t E
the adj^t $\angle^s$ CEA, CEB are tog. = two r^t $\angle^s$: (I. 13)
but, *from above*, $\angle^s$ AEC, AED = two r^t $\angle^s$,
∴ $\angle^s$ AEC, AED are tog. = CEA, CEB, (*axiom* 1)
take away the common ∠ AEC,
∴ rem^g ∠ AED = rem^g ∠ CEB. (*axiom* 3)
In the same manner it may be shown that
∠ AEC = DEB.
Wherefore, if two straight lines, &c.

Q. E. D.

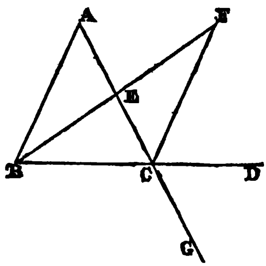
COR. I. From this it is manifest that if two straight lines cut each other, the angles which they make at the point where they cut, are together equal to four right angles.

COR. II. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP. XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior and opposite angles.

Let ABC be a $\triangle$, and let side BC be prod[d] to D:
Then ext[r] $\angle$ ACD shall be $>$ either of the int[r] opp. $\angle$[s] CBA, CAB.



Bisect AC in E (I. 10) and join BE;
prod. BE to F, making EF = BE, (I. 3); and join FC.
Then, in the $\triangle$[s] AEB, CEF,
$\because$ AE = EC and BE = EF, (*constr.*)
we have two sides AE, EB = two sides CE, EF, ea. to ea.,
and $\angle$ AEB = $\angle$ CEF, (I. 15)
$\therefore$ the $\triangle$[s] are equal in all respects, (I. 4).
Conseq. $\angle$ BAE = ECF,
but $\angle$ ACD $>$ ECF,
$\therefore$ also $\angle$ ACD $>$ BAC.
In the same manner, if side BC be bisected,
and AC be prod[d] to G, it may be demonstrated
that $\angle$ BCG $>$ ABC,
but $\angle$ BCG = ACD, (I. 15)
$\therefore$ also $\angle$ ACD $>$ ABC.

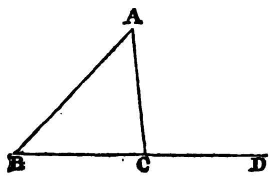
Wherefore, $\angle$ ACD has been shown to be $>$ either of the int[r] opp. $\angle$[s] CBA, CAB.

Q. E. D.

PROP. XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be a $\triangle$: any two of its angles, ABC, ACB *for instance*, are tog. < two r^t $\angle$^s.



Prod. BC to D.

Now, ext^r $\angle$ ACD being > int^r, opp. $\angle$ ABC (I. 16)

add $\angle$ ACB to each,

then $\angle$^s ACD, ACB are tog. > ABC, ACB,

but $\angle$^s ACD, ACB are tog. = two r^t $\angle$^s (I. 13)

$\therefore$ $\angle$^s ABC, ACB are tog. < two r^t $\angle$^s.

In the same manner it may be proved that *any other two angles* of the $\triangle$ are tog. < two r^t $\angle$^s.

Wherefore, any two angles of a triangle, &c.

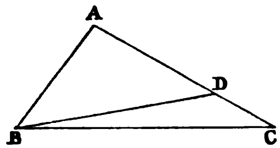
Q. E. D.

PROP. XVIII. THEOREM.

The greater side of a triangle is opposite to the greater angle.

In $\triangle$ ABC, let side AC be $>$ AB,

then $\angle$ ABC shall be $>$ ACB.



From AC cut off AD $=$ AB (I. 3) and join BD.

Now $\because$ AB $=$ AD (*constr.*)

$\therefore$ $\angle$ ABD $=$ ADB (I. 5)

but $\angle$ ADB $>$ DCB or ACB, (I. 16)

$\therefore$ $\angle$ ABD $>$ ACB,

à fortiori $\therefore$ $\angle$ ABC $>$ ACB.

Wherefore, the greater side of a triangle, &c.

Q. E. D.

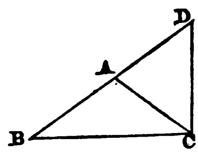
PROP. XIX. THEOREM.

The greater angle of a triangle is opposite to the greater side.

PROP. XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a $\triangle$: *any two* of its sides, BA, AC *for instance*, shall be tog. $>$ the third side, BC.



Prod. BA to D, making AD $=$ AC, and join CD.

Now, in $\triangle$ ACD, $\because$ AD $=$ AC (*constr.*)

$\therefore$ $\angle$ ADC $=$ ACD; (I. 5)

but $\angle$ BCD $>$ ACD,

$\therefore$ $\angle$ BCD $>$ ADC or BDC.

But the greater angle of a triangle is opposite to the greater side (I. 19).

$\therefore$ in $\triangle$ BCD, side BD $>$ BC,

but BA, AC are tog. $=$ BD (for AC $=$ AD)

$\therefore$ BA, AC are tog. $>$ BC.

In the same manner it may be proved that *any other two sides* of the $\triangle$ are tog. $>$ the third side.

Wherefore, any two sides of a triangle, &c.

Q. E. D.

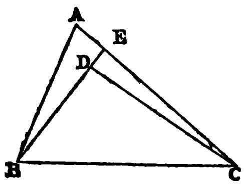
PROP. XXI. THEOREM.

If from the ends of a side of a triangle two straight lines be drawn to a point within the triangle, these shall together be less than the other two sides of the triangle, but shall contain a greater angle.

From B, C, the ends of side BC of △ ABC, let two st. lines BD, CD be drawn to p[t] D within △ ABC.

Then BD, DC shall tog. be < BA, AC,

but ∠ BDC shall be > BAC.



Prod. BD to E.

Now, any two sides of a △ being tog. > the third side, (I. 20)

two sides BA, AE are tog. > BE,

add EC to each,

then, we have BA, AC tog. > BE, EC; (*axiom* 4)

but, again, two sides DE, EC, of △ DEC, are tog. > DC, (I. 20)

add BD to each,

then, we have BE, EC tog. > BD, DC, (*axiom* 4)

but, *from above*, BA, AC are tog. > BE, EC,

à fortiori ∴ BA, AC are tog. > BD, DC,

or BD, DC are tog. < BA, AC. . . q. e. d.

Also, the ext[r] ∠ of a △ being > either of the int[r] opp. ∠[s], (I. 16)

ext[r] ∠ BDC > int[r] opp. ∠ DEC or BEC

and ext[r] ∠ BEC > int[r] opp. ∠ BAE or BAC,

à fortiori ∴ ∠ BDC > BAC.

Wherefore, if from the ends of a side of a triangle, &c.

Q. E. D.

PROP. XXII. PROBLEM.

To make a triangle whose sides shall be equal to three given straight lines, any two of which are together greater than the third.

PROP. XXIII. PROBLEM.

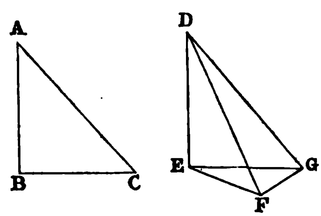
At a given point in a given straight line to make a rectilineal angle equal to a given rectilineal angle.

PROP. XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let Δ^s ABC, DEF have the two sides AB, AC = the two sides DE, DF, ea. to ea., viz., AB = DE, AC = DF; but let $\angle$ BAC be > angle EDF.

Then base BC shall be > base EF.



Of the two sides DE, DF, let DE be that wh. is not > the other.

At p[t] D, in st. line DE, make $\angle$ EDG = BAC, (I. 23) and make DG = AC or DF. (I. 3) Join EG, GF.

Now, in $\triangle^s$ ABC, DEG, we have

two sides BA, AC = ED, DG, ea. to ea.,

and $\angle$ BAC = EDG, (*constr.*)

$\therefore$ base BC = EG. (I. 4)

Again, $\because$ DF = DG (*constr.*)

$\therefore$ $\angle$ DFG = DGF, (I. 5)

but $\angle$ DGF > EGF, (*axiom* 9)

$\therefore$ $\angle$ DFG > EGF,

à fortiori $\therefore$ $\angle$ EFG > EGF.

But the greater $\angle$ of a $\triangle$ is opposite to the greater side (I. 19).

$\therefore$ in $\triangle$ EFG, side EG > EF,

but, *from above*, EG = BC

$\therefore$ BC > EF.

Wherefore, if two triangles have two sides of the one, &c.

Q. E. D.

PROP. XXV. THEOREM.

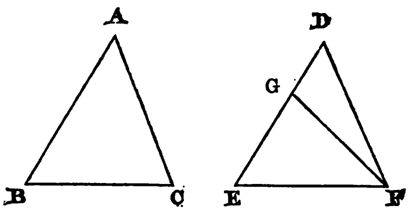
If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

PROP. XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side; viz., either the sides adjacent to the equal angles or sides opposite to equal angles in each triangle: then, shall the other sides be equal, each to each, and the third angle of the one to the third angle of the other; i.e. the triangles shall be equal in all respects.

Let $\triangle^s$ ABC, DEF have two $\angle^s$ of the one equal to two $\angle^s$ of the other, ea. to ea., viz., $\angle$ ABC = DEF, $\angle$ ACB = DFE; and, *first*, let BC = EF (*sides adj^t, in ea.* $\triangle$, *to both the angles named*).

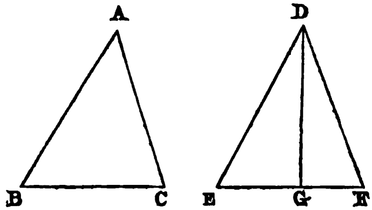
Then, the $\triangle^s$ shall be *equal in all respects*, viz., side AB = DE, side AC = DF, and $\angle$ BAC = EDF.



For, if AB be not = DE, suppose DE to be the greater; from ED cut off EG = AB, (I. 3) and join GF.

Now, AB being = GE, (*constr.*)
and BC being = EF, (*hyp.*)
∴ two sides AB, BC = two sides GE, EF, ea. to ea.,
and $\angle$ ABC = GEF, (*hyp.*)
∴ $\triangle^s$ ABC, GEF are *equal in all respects*, (I. 4)
conseq. $\angle$ ACB = GFE,
but $\angle$ ACB = DFE, (*hyp.*)
∴ $\angle$ GFE = DFE, (*axiom* 1)
wh. is impossible; (*axiom* 9)
∴ AB is not unequal to DE; *i.e.*, AB = DE.
Hence, in $\triangle^s$ ABC, DEF, we have
two sides AB, BC = DE, EF, ea. to ea.,
and inc^d $\angle$ ABC = inc^d $\angle$ DEF, (*hyp.*)
∴ $\triangle^s$ ABC, DEF are *equal in all respects*, (I. 4)
conseq. side AC = DF, and $\angle$ BAC = EDF.... q. e. d.

Secondly, ∠ˢ ABC, ACB being = DEF, DFE, ea. to ea., let AB = DE (*sides opp. to an equal* ∠ *in each* △), then, the remᵍ parts of the △ˢ shall be equal, ea. to ea., as before.



For, if BC be not = EF, suppose EF to be the greater; from EF cut off EG = BC, (I. 3) and join DG.

Now, AB being = DE, (*hyp.*)
and BC being = EG, (*constr.*)
∴ two sides AB, BC = two sides DE, EG, ea. to ea.,
and ∠ ABC = DEG, (*hyp.*)
∴ △ˢ ABC, DEG are *equal in all respects*, (I. 4)
conseq. ∠ ACB = DGE,
but ∠ ACB = DFE, (*hyp.*)
∴ ∠ DGE = DFE, (*axiom* 1)

i.e., the extʳ ∠ of △ DGF = an intʳ opp. ∠, wh. is impossible; (I. 16)

∴ BC is not unequal to EF, *i.e.*, BC = EF.
Hence, in △ˢ ABC, DEF, we have
the two sides AB, BC = DE, EF, ea. to ea.,
and ∠ ABC = DEF, (*hyp.*)
∴ △ˢ ABC, DEF are *equal in all respects*, (I. 4)
conseq. side AC = DF, and ∠ BAC = EDF.

Wherefore, if two triangles have two angles of the one, &c.

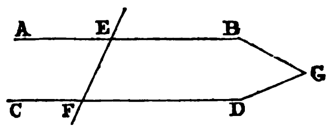
Q. E. D.

PROP. XXVII. THEOREM.

If a straight line, falling on two other straight lines, make the alternate angles equal to each other; these two straight lines shall be parallel.

Let the st. line EF, which falls upon the two st. lines AB, CD, make ∠ AEF = alt. ∠ EFD,

Then AB shall be ∥ CD.



For if AB be not ∥ CD,

then AB, CD being prod[d] will meet either towards A and C, or towards B and D;

Let them be prod[d] and meet, if possible, towards B and D in p[t] G,

then GEF is a △;

∴ its ext[r] ∠ AEF > int[r] opp. ∠ EFG, (I. 16)

but ∠ AEF = EFG, (*hyp.*)

∴ ∠[s] AEF and EFG are both equal and unequal,

which is absurd;

∴ AB, CD being prod[d] do not meet towards B and D:

In like manner it may be shown that they do not meet towards A and C;

but those st. lines, which, being prod[d] ever so far both ways, never meet, are parallel, (*def.*)

∴ AB ∥ CD.

Wherefore, if a straight line, &c.

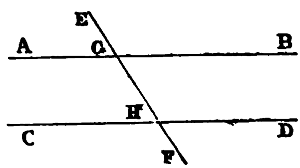
Q. E. D.

PROP. XXVIII. THEOREM.

If a straight line, falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line; or make the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the st. line EF, wh. falls upon the two st. lines AB, CD make the ext^r $\angle$ EGB $=$ int^r opp. $\angle$ GHD upon the same side; or make the two int^r $\angle^s$ BGH, GHD on the same side together $=$ two r^t $\angle^s$.

Then AB shall be $\parallel$ CD.



First, ext^r $\angle$ EGB being $=$ GHD (*hyp.*)
and $\angle$ EGB being $=$ AGH (I. 15)
$\therefore$ $\angle$ AGH $=$ alt. $\angle$ GHD (*axiom* 1)
$\therefore$ AB $\parallel$ CD (I. 27) . . . q. e. d.

Secondly, $\angle^s$ BGH, GHD being tog. $=$ two r^t $\angle^s$, (*hyp.*)
and $\angle^s$ AGH, BGH being also tog. $=$ two r^t $\angle^s$, (I. 13)
$\therefore$ $\angle^s$ AGH, BGH are tog. $=$ BGH, GHD; (*axiom* 1)
take away the common $\angle$ BGH,
then $\angle$ AGH $=$ alt. $\angle$ GHD,
$\therefore$ AB $\parallel$ CD. (I. 27)

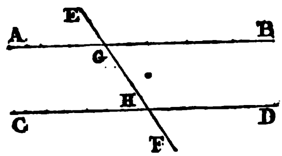
Wherefore, if a straight line, &c.

Q. E. D.

PROP. XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the st. line EF fall upon the ||^s AB, CD,
then ∠ AGH shall = alt. ∠ GHD,
ext^r ∠ EGB shall = int^r opp. ∠ GHD on the same side of EF,
and the two int^r ∠^{s} BGH, GHD shall be tog. = two r^t ∠^{s}.



For, if ∠ AGH be not = alt. ∠ GHD,
let ∠ AGH be > GHD,
add to each ∠ BGH,
then ∠^{s} AGH, BGH are tog. > BGH, GHD; (*axiom* 4)
but ∠^{s} AGH, BGH are tog. = two r^t ∠^{s}, (I. 13)
∴ ∠^{s} BGH, GHD are tog. < two r^t ∠^{s};
but those st. lines, wh. with another st. line falling upon them make the int^r ∠^{s} on the same side less than two r^t ∠^{s}, will meet together, if continually prod^d; (*axiom* 12)
but AB and CD can never meet, *being parallel by hyp.;*
∴ ∠ AGH = alt. ∠ GHD: . . . q. e. d.
but ∠ AGH = EGB, (I. 15)
∴ ext^r ∠ EGB = int^r opp. ∠ GHD
on same side of EF; (*axiom* 2) . . . q. e. d.
add to each ∠ BGH,
∴ ∠^{s} EGB, BGH are tog. = BGH, GHD, (*axiom* 1)
but ∠^{s} EGB, BGH are tog. = two r^t ∠^{s}, (I. 13)
∴ two int^r ∠^{s} BGH, GHD are tog. = two r^t ∠^{s}. (*axiom* 1)
Wherefore, if a straight line, &c.

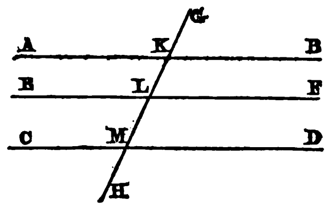
Q. E. D.

PROP. XXX. THEOREM.

Straight lines, which are parallel to the same straight line, are parallel to one another.

Let the st. lines AB, CD be each of them $\parallel$ EF.

Then AB shall be $\parallel$ CD.



Let st. line GH cut AB, EF, CD in p^ts^ K, L, M.

Then $\because$ GH cuts $\parallel^s$ AB, EF,

$\therefore$ $\angle$ AKL = alt. $\angle$ KLF: (I. 29)

Again $\because$ GH cuts $\parallel^s$ EF, CD,

$\therefore$ ext^r $\angle$ KLF = int^r opp. $\angle$ LMD; (I. 29)

but, *from above*, $\angle$ AKL = KLF,

$\therefore$ $\angle$ AKL = KMD;

and these are alt. $\angle^s$ with respect to st. lines AB, CD,

$\therefore$ AB $\parallel$ CD. (I. 27)

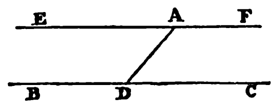
Wherefore, straight lines, which are, &c.

Q. E. D.

PROP. XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given p[t], and BC the given st. line.
It is req[d] to draw through A a st. line || BC.



In BC take any p[t] D, and join AD;
at the p[t] A in DA make ∠ DAE = ADC, (I. 23)
and prod. EA to F:
then EF || BC.
For, ∵ AD meets EF, BC,
and ∠ EAD = alt. ∠ ADC (*constr.*)
∴ EF || BC. (I. 27)

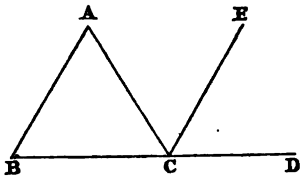
Wherefore, through the given p[t] A, a st. line EF has been drawn parallel to the given st. line BC.

Q. E. F.

PROP. XXXII. THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior opposite angles: and the three interior angles of every triangle are together equal to two right angles.

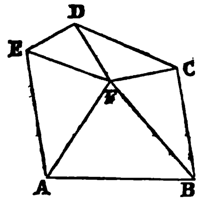
Let side BC of △ ABC be prod^d to D,
then ext^r ∠ ACD shall = two int^r opp. ∠^s CAB, ABC,
and the three int^r ∠^s ABC, BCA, CAB shall = two r^t ∠^s.



Through p^t C draw CE ‖ BA. (I. 31)
Then ∵ AC meets ‖^s BA, CE,
∴ ∠ BAC = alt. ∠ ACE; (I. 29)
and ∵ BD meets ‖^s BA, CE,
∴ ext^r ∠ ECD = int^r opp. ∠ ABC; (I. 29)
but, *from above*, ∠ ACE = BAC,
∴ whole ext^r ∠ ACD = two int^r opp. ∠^s CBA, BAC: (*ax.* 2)... q.e.d.
add to each ∠ ACB,
∴ ∠^s ACD, ACB = the three ∠^s CBA, BAC, ACB; (*axiom* 2)
but ∠^s ACD, ACB are tog. = two r^t ∠^s, (I. 13)
∴ ∠^s CBA, BAC, ACB are tog. = two r^t ∠^s. (*axiom* 2)
Wherefore, if a side of a triangle, &c.

Q. E. D.

COR. I. All the int^r $\angle^s$ of any rect^l fig. tog. with four r^t $\angle^s$ are equal to twice as many r^t $\angle^s$ as the fig. has sides.



For any rect^l fig. ABCDE can be div^d into as many Δ^s as the fig. has sides, by drawing st. lines from any p^t F within the fig. to each of its $\angle^s$.

Then $\because$ the three int^r $\angle^s$ of a Δ = two r^t $\angle^s$,

and there are as many Δ^s, as fig. has sides;

$\therefore$ all the $\angle^s$ of these Δ^s = twice as many r^t $\angle^s$ as fig. has sides;

but the same $\angle^s$ of these Δ^s = the int^r $\angle^s$ of fig. tog. with $\angle^s$ at F,

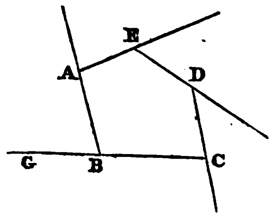
and $\angle^s$ at F = four r^t $\angle^s$, (I. 15, Cor. 2)

$\therefore$ the same $\angle^s$ of these Δ^s = the int^r $\angle^s$ of fig. tog. with four r^t $\angle^s$;

but, *from above*, the $\angle^s$ of the Δ^s = twice as many r^t $\angle^s$ as fig. has sides,

$\therefore$ the int^r $\angle^s$ of fig. tog. with four r^t $\angle^s$ = twice as many r^t $\angle^s$ as fig. has sides.

COR. II. All the ext^r $\angle^s$ of any rect^l fig. made by prod^g the sides successively in the same direction are tog. = four r^t $\angle^s$.



Since every int^r $\angle$ ABC tog. with its adj^t ext^r $\angle$ ABG = two r^t $\angle^s$, (I. 13)

$\therefore$ all the int^r $\angle^s$ tog. with all the ext^r $\angle^s$ = twice as many r^t $\angle^s$ as fig. has sides;

but all the int^r $\angle^s$ tog. with four r^t $\angle^s$ = twice as many r^t $\angle^s$ as fig. has sides, (Cor. 1)

$\therefore$ all the int^r $\angle^s$ tog. with all the extr^r $\angle^s$ = all the int^r $\angle^s$ tog. with four r^t $\angle^s$,

take from these equals the int^r $\angle^s$,

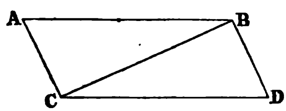
$\therefore$ all the ext^r $\angle^s$ = four r^t $\angle^s$. (*axiom* 2)

PROP. XXXIII. THEOREM.

The straight lines, which join the corresponding extremities of two equal and parallel straight lines, are also themselves equal and parallel.

Let AB, CD be equal and parallel st. lines, joined at the corresponding ext[ies] by st. lines AC and BD :

AC and BD shall be equal and parallel.



Join BC.

Now ∵ BC meets the ||[s] AB, CD,

∴ ∠ ABC = alt. ∠ BCD. (I. 29)

Hence, in △[s] ABC, BCD,

∵ AB = CD, and BC is common to both △[s],

we have two sides AB,BC = two sides DC, CB, ea. to ea.,

and ∠ ABC = DCB,

∴ the △[s] are equal in all respects : (I. 4)

conseq. AC = BD,

and ∠ ACB = CBD.

Also ∵ BC meets the st. lines AC, BD,

making ∠ ACB = alt. ∠ CBD,

∴ AC || BD : (I. 27)

and, *from above*, AC = BD.

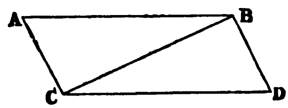
Wherefore, straight lines, which join, &c.

Q. E. D.

PROP. XXXIV. THEOREM.

The opposite sides and angles of parallelograms are equal to one another; and the diagonal bisects them, that is, divides them into two equal parts.

Let ACDB be a ▭m, and BC its diagonal:
the opp. sides and ∠s of the fig. shall be equal to one another, and the diag[l] shall bisect it.



For ∵ BC meets the ||s AB, CD,
∴ ∠ ABC = alt. ∠ BCD; (I. 29)
And ∵ BC meets the ||s AC, BD,
∴ ∠ ACB = alt. ∠ CBD. (I. 29)
Hence in △s ABC, CBD,
we have two ∠s ABC, ACB = two ∠s BCD, CBD, ea. to ea.,
and side BC is common to both,
∴ △s are equal in all respects, (I. 26)
conseq. fig. ACDB is bisected by BC, . . . q. e. d.
also AB = CD,
AC = BD,
and ∠ CAB = CDB:
and ∵ ∠ ABC = BCD, and ∠ CBD = ACB (*from above*)
∴ whole ∠ ABD = whole ∠ ACD (*axiom* 2)
∴ the opp. sides and ∠s of ▭m ACDB are equal to one another,
and it was shown to be bisected by its diag[l] BC.

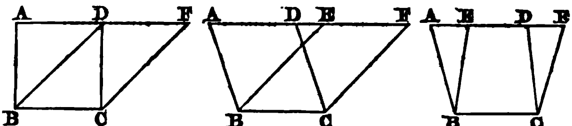
Q. E. D.

PROP. XXXV. THEOREM.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let $\square^{ms}$ ABCD, EBCF be upon the same base BC, and between same $\|^{s}$ AF, BC;

$\square^{m}$ ABCD shall be $=$ $\square^{m}$ EBCF.



(1) (2) (3)

First, in the case where the sides of the $\square^{ms}$, opp. to the base, terminate in the same p^{t}, as at D in fig. (1),

$$\left.\begin{array}{r}\square^{m}\ \text{ABCD is double of }\triangle\text{ DBC,}\\ \text{and }\square^{m}\ \text{DBCF is double of }\triangle\text{ DBC,}\end{array}\right\}\text{(I. 34)}$$

$\therefore$ $\square^{m}$ ABCD $=$ $\square^{m}$ DBCF. (*axiom* 6)

Secondly, when the sides AD, EF do not terminate in same p^{t}:

then $\because$ AD $=$ BC and EF $=$ BC, (I. 34)

$\therefore$ AD $=$ EF; (*axiom* 1)

and adding DE to each, fig. (2),

or subtracting DE from each, fig. (3)

whole or rem^{r} AE $=$ whole or rem^{r} DF.

Hence, in $\triangle^{s}$ FDC, EAB

$\because$ FD $=$ EA, and DC $=$ AB, (I. 34)

we have two sides FD, DC $=$ two sides EA, AB, ea. to ea.,

and $\angle$ FDC $=$ EAB, (I. 29)

$\therefore$ $\triangle$ FDC $=$ $\triangle$ EAB. (I. 4)

Now from trapezium ABCF taking away $\triangle$ FDC, $\square^{m}$ ABCD remains,

and from same trapezium ABCF taking away $\triangle$ EAB, $\square^{m}$ EBCF remains,

$\therefore$ $\square^{m}$ ABCD $=$ $\square^{m}$ EBCF. (*axiom* 3)

Wherefore, parallelograms upon the same base, &c.

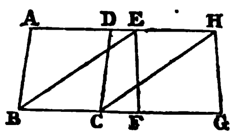
Q. E. D.

PROP. XXXVI. THEOREM.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be $\square^{ms}$ upon equal bases BC, FG, and between the same $\|^{s}$ AH, BG;

$\square^{m}$ ABCD shall be = $\square^{m}$ EFGH.



Join BE, CH.

Now ∵ BC = FG (*hyp.*) and FG = EH, (I. 34)

∴ BC = EH, (*axiom* 1)

and BC ∥ EH; (*hyp.*)

but BC and EH are joined at the corresponding ext^{ies} by EB and HC,

∴ EB and HC are equal and parallel: (I. 33)

hence EBCH is a $\square^{m}$: (*def.*)

and $\square^{ms}$ ABCD, EBCH are on same base BC, and between same $\|^{s}$ BC, AH.

∴ $\square^{m}$ ABCD = $\square^{m}$ EBCH. (I. 35)

For a similar reason $\square^{m}$ EFGH = $\square^{m}$ EBCH,

∴ $\square^{m}$ ABCD = $\square^{m}$ EFGH. (*axiom* 1)

Wherefore, parallelograms upon equal, &c.

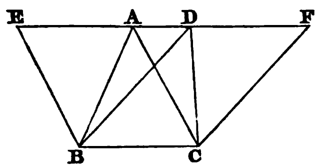
Q. E. D.

PROP. XXXVII. THEOREM.

Triangles upon the same base and between the same parallels are equal to one another.

Let $\triangle^s$ ABC, DBC be upon the same base BC and between the same $\|^s$ AD, BC :

these $\triangle^s$ shall be equal.



Thro' B draw BE $\|$ AC, meeting DA prod^d in E ;

thro' C draw CF $\|$ BD, meeting AD prod^d in F ;

then, figs. ACBE, DBCF are $\square^{ms}$, (*def.*)

and they are on the same base BC and between the same $\|^s$ EF, BC,

$\therefore$ these $\square^{ms}$ are equal. (I. 35)

$$\left.\begin{array}{l}\text{But } \triangle \text{ ABC is half } \square^{m} \text{ ACBE,} \\ \text{and } \triangle \text{ DBC is half } \square^{m} \text{ DBCF,}\end{array}\right\} \text{(I. 34)}$$

and the halves of equal things are themselves equal, (*axiom* 7)

$\therefore$ $\triangle$ ABC = $\triangle$ DBC.

Wherefore, triangles upon the same base, &c.

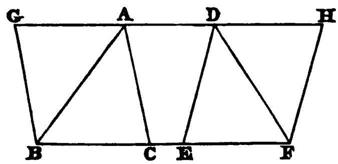
Q. E. D.

PROP. XXXVIII. THEOREM.

Triangles upon equal bases and between the same parallels are equal to one another.

Let $\triangle^s$ ABC, DEF be upon equal bases BC, EF and between the same $\|^s$ AD, BF :

these $\triangle^s$ shall be equal.



Thro' B draw BG $\|$ AC, meeting DA prod[d] in G ;

thro' F draw FH $\|$ ED, meeting AD prod[d] in H ;

then figs. ACBG, DEFH are $\square^{ms}$, (*def.*)

and they are on equal bases BC, EF and between the same $\|^s$ GH, BF,

$\therefore$ these $\square^{ms}$ are equal. (I. 36)

$$\left.\begin{array}{l}\text{But } \triangle \text{ ABC is half } \square^{m} \text{ ACBG,}\\ \text{and } \triangle \text{ DEF is half } \square^{m} \text{ DEFH,}\end{array}\right\}\text{(I. 34)}$$

and the halves of equal things are themselves equal, (*axiom* 7)

$\therefore$ $\triangle$ ABC = $\triangle$ DEF.

Wherefore, triangles upon equal bases, &c.

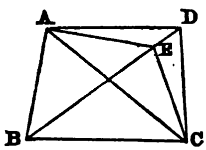
Q. E. D.

PROP. XXXIX. THEOREM.

Equal triangles upon the same base and upon the same side of it are between the same parallels.

Let the equal $\triangle^s$ ABC, DBC be upon the same side of the same base BC:

the st. line AD, joining their vertices shall be || BC.



For, if not, suppose AE || BC and join EC;

then $\triangle^s$ ABC, EBC are upon the same base BC and between the same $||^s$ AE, BC;

$\therefore$ $\triangle$ ABC = $\triangle$ EBC: (I. 37)

but $\triangle$ ABC = $\triangle$ DBC, (*hyp.*)

$\therefore$ $\triangle$ EBC = $\triangle$ DBC, (*axiom* 1)

wh. is impossible;

$\therefore$ AE cannot be || BC, and in the same manner it may be shown that no other st. line than AD can be || BC;

$\therefore$ AD *is* || BC.

Wherefore, equal triangles upon the same base, &c.

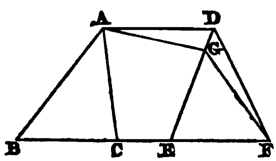
Q. E. D.

PROP. XL. THEOREM.

Equal triangles upon equal bases, in the same straight line and on the same side of it, are between the same parallels.

Let the equal Δ^s ABC, DEF be upon equal bases BC, EF, these bases being in one st. line BF, and the Δ^s being on the same side of BF:

the st. line AD, joining their vertices, shall be $\parallel$ BF.



For, if not, suppose AG $\parallel$ BF, and join GF;
then, Δ^s ABC, GEF are upon equal bases BC, EF, and between the same $\parallel^s$ AG, BF;

$\therefore$ Δ ABC $=$ Δ GEF: (I. 38)

but Δ ABC $=$ Δ DEF, (*hyp.*)

$\therefore$ Δ GEF $=$ Δ DEF, (*axiom* 1)

wh. is impossible;

$\therefore$ AG cannot be $\parallel$ BF, and in the same manner it may be shown that no other st. line than AD can be $\parallel$ BF;

AD *is* $\parallel$ BF.

Wherefore, equal triangles, &c.

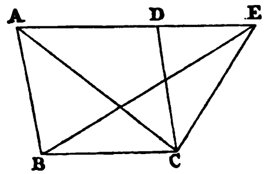
Q. E. D.

PROP. XLI. THEOREM.

If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let ▭ᵐ ABCD and △ EBC be upon the same base BC and between the same ||ˢ AE, BC:

▭ᵐ ABCD shall be double of △ EBC.



Join AC.

Now △ˢ ABC, EBC are upon the same base BC and between the same ||ˢ AE, BC,

∴ △ ABC = △ EBC: (I. 37)

but ▭ᵐ ABCD is double of △ ABC, (I. 34)

∴ ▭ᵐ ABCD is double of △ EBC.

Wherefore, if a parallelogram and a triangle, &c.

Q. E. D.

PROP. XLII. PROBLEM.

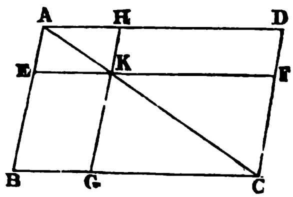
To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

PROP. XLIII. THEOREM.

The complements of the parallelograms, which are about the diagonal of a parallelogram, are equal to one another.

Let ABCD be a $\square^{m}$, AC its diagonal, EH, GF $\square^{ms}$ *about* the diagonal AC, and EG, HF the complements.

Comp^{t} EG shall $=$ comp^{t} HF.



For, AC being the diagonal of the three $\square^{ms}$ ABCD, EH, GF, it bisects them all;

$$\therefore \left.\begin{array}{l} \triangle\ ABC = \triangle\ ADC, \\ \triangle\ AEK = \triangle\ AHK, \\ \triangle\ KGC = \triangle\ KFC: \end{array}\right\} \text{(I. 34)}$$

conseq. $\triangle^{s}$ AEK, KGC are tog. $=$ $\triangle^{s}$ AHK, KFC. (*axiom* 2)

From $\triangle$ ABC take $\triangle^{s}$ AEK, KGC and comp^{t} EG remains;

from $\triangle$ ADC take $\triangle^{s}$ AHK, KFC and comp^{t} HF remains;

$\therefore$ comp^{t} EG $=$ comp^{t} HF. (*axiom* 3)

Wherefore, the complements, &c.

Q. E. D.

PROP. XLIV. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

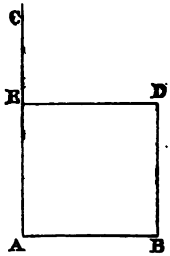
PROP. XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

PROP. XLVI. PROBLEM.

To describe a square upon a given straight line.

Let it be req^d to desc. a square upon a given st. line AB.



At p^t A draw AC $\perp$ AB; (I. 11)
from AC cut off AE = AB; (I. 3)

$\left.\begin{array}{r}\text{thro' E draw ED} \parallel \text{AB,} \\ \text{and thro' B draw BD} \parallel \text{AE:}\end{array}\right\}$ (I. 31)

then fig. ABDE is a $\square^m$: (*def.*)

$\left.\begin{array}{r}\therefore \text{AB} = \text{ED,} \\ \text{and AE} = \text{BD,}\end{array}\right\}$ (I. 34)

but AB = AE, (*constr.*)
$\therefore$ AB = AE = ED = BD:
conseq. fig. ABDE has all its sides equal.
Again, $\because$ st. line AC meets the $\parallel^s$ AB, ED,
$\therefore$ two int^r $\angle^s$ AED, EAB are tog. = two r^t $\angle^s$; (I. 29)
but EAB is a r^t $\angle$, (*constr.*)
$\therefore$ AED is also a r^t $\angle$;
and the opp. $\angle^s$ of $\square^{ms}$ are equal to one another, (I. 34)
$\therefore$ fig. ABDE has all its angles r^t $\angle^s$,
and it was proved to be equilateral.

Wherefore, it is a square (*def.*), and it is desc^d upon the given st. line AB.

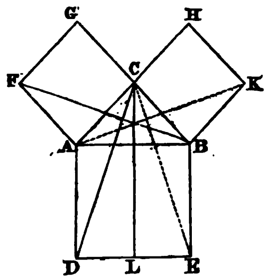
Q. E. F.

Cor. Hence every parallelogram, that has one right angle, has all its angles right angles.

PROP. XLVII. THEOREM.

In a right-angled triangle the square described upon the side opposite the right angle is equal to the squares described upon the sides which contain the right angle.

Let ABC be a r^{t} $\angle^{d}$ $\triangle$, having the r^{t} $\angle$ ACB: then the square desc^{d} upon AB shall be equal to the squares desc^{d} upon AC and BC.



$$\left.\begin{array}{l}\text{Upon AB desc. sq. ADEB;}\\ \text{upon AC desc. sq. AFGC;}\\ \text{upon BC desc. sq. CBKH;}\end{array}\right\}\text{(I. 46)}$$

Thro' C draw CL $\parallel$ AD;

Join FB, CD.

Now, ACB being a r^{t} $\angle$, (*hyp.*)

and ACG being a r^{t} $\angle$, (*constr.*)

$\therefore$ at p^{t} C in st. line AC, the two st. lines BC, GC meet together making the adj^{t} $\angle^{s}$ tog. = two r^{t} $\angle^{s}$;

$\therefore$ BCG is a st. line. (I. 14)

In the same manner it may be shown that ACH is a st. line.

Now, ea. of the ∠s FAC, BAD being a r^t ∠, (*constr.*)

add ∠ CAB to each,

then whole ∠ FAB = whole ∠ CAD; (*axiom* 2)

$$\left.\begin{array}{l}\text{and FA being} = \text{CA,}\\ \text{and AB being} = \text{AD,}\end{array}\right\}\textit{sides of squares}$$

we have the two sides FA, AB = two sides CA, AD, ea. to ea.

and, *from above*, inc^d ∠ FAB = inc^d ∠ CAD,

∴ △ CAD = △ FAB. (I. 4)

And, △ CAD and ▭^m AL are upon the same base AD and between the same ||s AD, CL,

∴ ▭^m AL is double of △ CAD; (I. 41)

also, △ FAB and ▭^m or sq. AG are upon the same base FA, and between the same ||s FA, GB,

∴ sq. AG is double of △ FAB;

and the doubles of equal things are themselves equal, (*axiom* 6)

∴ ▭^m AL = sq. AG.

In the same manner, by joining CE, AK, it may be proved that ▭^m BL = sq. BH,

∴ ▭^ms AL, BL are tog. = sq^s AG, BH;

But the two ▭^ms AL, BL tog. make up the whole fig. ADEB, wh. is the sq. desc^d upon AB;

and AG, BH are the sq^s desc^d upon AC, BC;

∴ sq. desc^d upon AB = sq^s desc^d upon AC and BC.

Wherefore, in a right-angled triangle, &c.

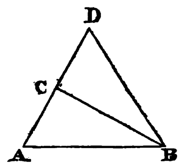
Q. E. D.

NOTE.—The phrase "sq. desc^d upon AB" is usually written more briefly "sq. of AB."

PROP. XLVIII. THEOREM.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides, the angle contained by these two sides is a right angle.

Let ABC be a $\triangle$, and let sq. of AB = sq[s] of AC and BC: then ACB shall be a r[t] $\angle$.



At p[t] C draw CD $\perp$ CB, (I. 11)
making CD = AC, and join BD.
Now, $\because$ AC = CD; (*constr.*)
$\therefore$ sq. of AC = sq. of CD,
add sq. of CB to each,
then sq[s] of AC and CB are tog. = sq[s] of DC and CB:
But sq. of AB = sq[s] of AC and CB, (*hyp.*)
and sq. of BD = sq[s] of DC and CB, (I. 47)
$\therefore$ sq. of AB = sq. of DB,
$\therefore$ AB = DB.

Hence, in $\triangle$[s] ACB, DCB, we have two sides AC, CB = two sides DC, CB, ea. to ea., and base AB = base DB,

$\therefore$ $\angle$ ACB = DCB: (I. 8.)
but DCB is a r[t] $\angle$ (*constr.*)
$\therefore$ ACB is a r[t] $\angle$.

Wherefore, if the square described, &c.

Q. E. D.

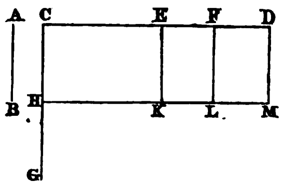
BOOK II.

PROP. I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.

Let AB, CD be two st. lines, CD being divided into *any n°* of parts in p^{ts} E, F, &c.:

Then the rect. $cont^d$ by AB and CD shall be = the $rect^s$ $cont^d$ by AB and CE, AB and EF, AB and FD.



At C draw CG ⊥ CD, (I. 11) and from it cut off CH = AB; (I. 3) thro' H draw HM ∥ CD, and thro' E, F, D draw EK, FL, DM each ∥ CG. (I. 31)

Now, whole fig. CM = fig^s CK, EL, FM,

but fig. CM = rect. $cont^d$ by AB and CD,

for it *is* $cont^d$ by CH and CD, of which CH = AB (*constr.*)

and fig^s CK, EL, FM are $resp^{ly}$ = $rect^s$ $cont^d$ by AB and CE, AB and EF, AB and FD,

for each of the lines EK, FL, DM = CH = AB; (I. 34)

∴ rect. AB·CD = $rect^s$ AB·CE, AB·EF, AB·FD.

Wherefore, if there be two straight lines, &c.

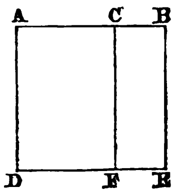
Q. E. D.

PROP. II. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole line and each of the parts are together equal to the square of the whole line.

Let st. line AB be div[d] into any two parts in p[t] C.

Then the rect. cont[d] by AB and AC, tog. with rect. cont[d] by AB and BC shall be = sq. of AB.



Upon AB desc. the sq. ADEB; (I. 46)

and thro' C draw CF || AD. (I. 31)

Now, the two fig[s] AF, CE are tog. = the whole fig. AE:

but AF = the rect. AB·AC,

for it is cont[d] by AD and AC, of wh. AD = AB (*sides of a square*);

and CE = the rect. AB·BC,

for it is cont[d] by BE and BC, of wh. BE = AB;

and AE is the sq. of AB: (*constr.*)

∴ rect[s]. AB·AC, AB·BC are tog. = sq. of AB.

Wherefore, if a straight line be divided, &c.

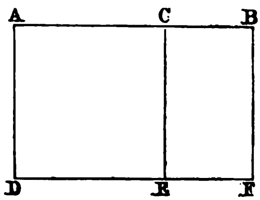
Q. E. D.

PROP. III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let st. line AB be div[d] into any two parts in p[t] C.

Then the rect. AB·AC shall be = rect. AC·BC tog. with the sq. of AC.



Upon AC desc. the square ADEC; (I. 46)
prod. DE to F, and thro' B draw BF || CE. (I. 31)
Now, whole fig. AF = the two fig[s] CF, AE;
but AF = the rect. AB·AC,
for it is cont[d] by AB and AD, of wh. AD = AC, (*sides of a square*);
CF = the rect. AC·CB,
for it is cont[d] by CE and CB, of which CE = AC;
and AE is the sq. of AC; (*constr.*)
∴ rect. AB·AC = rect. AC·CB tog. with the sq. of AC.

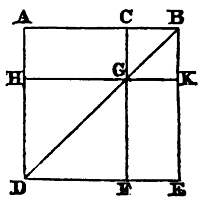
Wherefore, if a straight line be divided, &c.

Q. E. D.

PROP. IV. THEOREM.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let st. line AB be div[d] into any two parts in p[t] C.
Then the sq. of AB shall be = the sq[s] of AC and CB tog. with twice the rect. AC·CB.



Upon AB desc. the sq. ADEB (I. 46), and join BD;
thro' C draw CF ∥ AD, and thro' G draw HK ∥ AB. (I. 31)
Now, ∵ AD = AB, (*sides of a square*)
∴ ∠ ADB = ABD; (I. 5)
and ∵ st. line BD cuts the ∥[s] AD, CF,
∴ ext[r] ∠ CGB = int[r] opp. ∠ ADB; (I. 29)
but, *from above*, ∠ ADB = ABD,
∴ ∠ CGB = CBG;
and ∴ side CB = side CG; (I. 6)
and fig. CK being a ▭[m], (*constr.*)
∴ CB = CG = GK = BK; (I. 34)
conseq. fig. CK has all its sides equal:
and since ∠ KBC is a r[t] ∠, (*constr.*)
∴ fig. CK is also rectangular; (I. 46, Cor.)
conseq. CK is a square, and it is desc[d] on CB.

Sim^{ly} it may be shown that HF is a square, and it is upon the side HG, wh. = AC, (I. 34)

∴ fig. HF = the sq. of AC :

conseq. the fig^{s} HF and CK = the sq^{s} of AC and CB.

Also the comp^{t} AG = the comp^{t} GE, (I. 43)

and fig. AG = rect. AC·CB, (for CG = CB)

∴ the two comp^{ts} AG, GE are tog. = twice the rect. AC·CB.

Wherefore the four fig^{s} HF, CK, AG, GE are = the sq^{s} of AC and CB tog. with twice the rect. AC·CB;
but the four fig^{s} HF, CK, AG, GE make up the fig. ADEB, wh. is the sq. of AB, (*constr.*)
∴ the sq. of AB = sq^{s} of AC and CB tog. with twice the rect. AC·CB.

Wherefore, if a straight line be divided, &c.

Q. E. D.

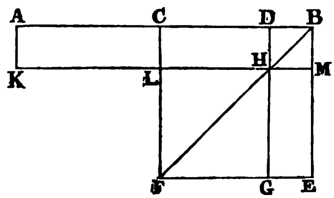
COR. From the demonstration, it is manifest that the parallelograms about the diagonal of a square are likewise squares.

PROP. V. THEOREM.

If a straight line be divided into two equal and also into two unequal parts, the rectangle contained by the unequal parts together with the square of the line between the points of section is equal to the square of half the line.

Let st. line AB be div[d] into two equal parts in C, and into two unequal parts in D.

Then the rect. AD·DB tog. with the sq. of CD shall be = the sq. of CB.



Upon CB desc. the square CFEB; (I. 46) and join BF;
thro' D draw DHG ‖ BE, (I. 31)
thro' H draw KLM ‖ AB,
and thro' A draw AK ‖ CF.
Now, comp[t] CH being = comp[t] HE, (I. 43)
add fig. DM to each,
then fig. CM = fig. DE:
but ∵ AC = CB,
∴ fig. CM = fig. AL; (I. 36)
∴ fig. AL = fig. DE;

add fig. CH to each,

then whole fig. AH = the fig^s CH and DE;

but AH = rect. AD·DB, for DH = DB, (II. 4, Cor.)

and fig^s CH and DE are tog. = gnomon CMG,

∴ rect. AD·DB = gnomon CMG;

add to each the fig. LG, wh. = the sq. of CD, (II. 4, Cor.)

then rect. AD·DB tog. with sq. of CD = gnomon CMG tog. with fig. LG;

but gnomon CMG tog. with fig. LG make up fig. CFEB, wh. is the sq. of CB, (*constr.*)

∴ rect. AD·DB tog. with sq. of CD = sq. of CB.

Wherefore, if a straight line be divided, &c.

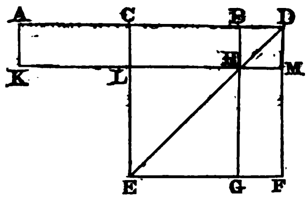
Q. E. D.

Cor. From this proposition, it is manifest that the difference of the squares of two unequal lines AC, CD is equal to the rectangle contained by their sum AD, and difference DB.

PROP. VI. THEOREM.

If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the st. line AB be bisected in C, and prod[d] to the p[t] D. Then the rect. AD·DB tog. with the sq. of CB, shall = the sq. of CD.



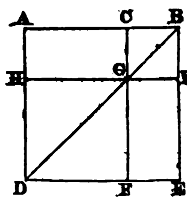
Upon CD desc. the sq. CEFD; (I. 46) and join DE;
thro' B draw BHG ∥ DF, (I. 31)
thro' H draw KLM ∥ AD,
and thro' A draw AK ∥ CE.
Now, comp[t] CH = comp[t] HF; (I. 43)
and, CB being = AC,
fig. CH = fig. AL; (I. 36)
∴ fig. AL = fig. HF:
add CM to each,
then whole fig. AM = gnomon CMG;
but AM = rect. AD·DB, for DM = DB, (II. 4, Cor.)
∴ gnomon CMG = rect. AD·DB:
add to each fig. LG, wh. = the sq. of CB; (II. 4, Cor.)
then rect. AD·DB tog. with the sq. of CB = gnomon CMG tog. with fig. LG;
but gnomon CMG tog. with fig. LG make up fig. CEFB, wh. is the sq. of CD, (*constr.*)
∴ rect. AD·DB tog. with sq. of CB = sq. of CD.
Wherefore, if a straight line, &c.

Q. E. D.

PROP. VII. THEOREM.

If a straight line be divided into any two parts, the squares of the whole line and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the st. line AB be div^d into any two parts in pt. C: Then the sq^s of AB and BC shall be tog. = twice rect. AB·BC together with sq. of AC.



Upon AB descr. the sq. ADEB, (I. 46) and join BD;
thro' C draw CGF ∥ BE, (I. 31)
and thro' G draw HK ∥ AB.
Now comp^t AG = comp^t GE, (I. 43)
add to each fig. CK,
then whole fig. AK = whole fig. CE;
hence fig^s AK, CE are tog. = twice the fig. AK;
but fig. AK = rect. AB·BC, for BK, = BC (II. 4, Cor.)
and fig^s AK, CE tog. make up gnomon AKF and fig. CK,
∴ gnomon AKF and fig. CK are tog. = twice rect. AB·BC;
add to each fig. HF, wh. = sq. of AC (II. 4, Cor.)
then gnomon AKF and fig^s CK, HF = twice rect. AB·BC tog. with sq. of AC;
but gnomon AKF tog. with fig^s CK, HF make up the two fig^s ADEB and CK, wh. are the sq^s of AB and BC,
∴ sq^s of AB and BC = twice rect. AB·BC tog. with sq. of AC.
Wherefore, if a straight line, &c.

Q. E. D.

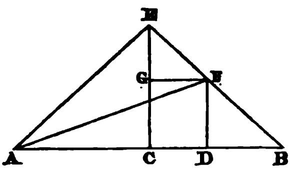
PROP. VIII. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line, which is made up of the whole and that part.

PROP. IX. THEOREM.

If a straight line be divided into two equal and also into two unequal parts, the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let the st. line AB be div^d into two equal parts in p^t C, and into two unequal parts in D;
The sq^s of AD and DB shall be tog. double of sq^s of AC and CD.



Thro' C draw CE $\perp$ AB, (I. 11)
making CE = AC or CB, and join EA, EB;
thro' D draw DF $\parallel$ CE, meeting EB in F; (I. 31)
thro' F draw FG $\parallel$ BA, and join AF.
Now, AC being = CE (*constr.*)
$\angle$ AEC = EAC; (I. 5)
and $\because$ ACE is a r^t $\angle$,
$\therefore$ $\angle^s$ AEC, EAC are tog. = a r^t $\angle$: (I. 32)
and since they are equal to one another, each of them is half a r^t $\angle$;
for the same reason each of $\angle^s$ CEB, EBC is half a r^t $\angle$,
$\therefore$ whole $\angle$ AEB is a r^t $\angle$.

Again, EGF is a r^{t} $\angle$, being = ECB (I. 29)
and, *from above*, GEF is half a r^{t} $\angle$,
$\therefore$ rem^{g} $\angle$ EFG is half a r^{t} $\angle$,
hence $\angle$ EFG = GEF,
$\therefore$ side EG = side GF. (I. 6)
Sim^{ly} it may be proved that FD = DB.
And $\because$ AC = CE,
$\therefore$ sq. of AC = sq. of CE,
$\therefore$ sq^{s} of AC and CE are tog. double of sq. of AC;
but sq. of AE = sq^{s} of AC and CE (I. 47)
$\therefore$ sq. of AE is double of sq. of AC:
and $\because$ EG = GF,
$\therefore$ sq. of EG = sq. of GF,
$\therefore$ sq^{s} of EG and GF are tog. double of sq. of GF,
but sq. of EF = sq^{s} of EG and GF, (I. 47)
$\therefore$ sq. of EF is double of sq. of GF;
and GF = CD, (I. 34)
$\therefore$ sq. of EF is double of sq. of CD:
but, *from above*, sq. of AE is double of sq. of AC,
$\therefore$ sq^{s} of AE and EF are tog. double of sq^{s} of AC and CD;
but sq. of AF = sq^{s} of AE and EF, (I. 47)
$\therefore$ sq. of AF is double of sq^{s} of AC and CD;
but, sq^{s} of AD and DF = sq. of AF, (I. 47)
$\therefore$ sq^{s} of AD and DF are tog. double of sq^{s} of AC and CD;
but, *from above*, DF = DB,
$\therefore$ sq^{s} of AD and DB are tog. double of sq^{s} of AC and CD.
Wherefore, if a straight line, &c.

Q. E. D.

PROP. X. THEOREM.

If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

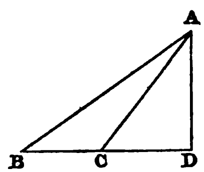
PROP. XI. PROBLEM.

To divide a straight line into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square of the other part.

PROP. XII. THEOREM.

In an obtuse-angled triangle, if a perpendicular be let fall from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is equal to the squares of the sides which contain it, together with twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted, outside the triangle, between the perpendicular and the obtuse angle.

Let BAC be an obtuse $\angle^d$ $\triangle$, having the obtuse $\angle$ ACB, and from p^t A let fall AD $\perp$ BC prod^d;
Then sq. of AB shall be = sq^s of AC and CB tog. with twice rect. BC·CD.



For, $\because$ BD is div^d into two parts in C,
$\therefore$ sq. of BD = sq^s of BC and CD tog. with twice rect. BC·CD, (II. 4)
add to each sq. of DA,
then sq^s of BD and DA = sq^s of BC, CD, and DA tog. with twice rect. BC·CD;
but sq^s of BD and DA = sq. of BA
and sq^s of CD and DA = sq. of CA } (I. 47)
$\therefore$ sq. of BA = sq^s of BC and CA tog. with twice rect. BC·CD.
Wherefore, in an obtuse-angled triangle, &c.

Q. E. D.

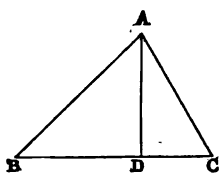
PROP. XIII. THEOREM.

In every triangle, the square of the side subtending either of the acute angles is equal to the squares of the sides containing that angle, less twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle, and the perpendicular let fall upon the side from the opposite angle.

Let ABC be *any* $\triangle$, $\angle$ B being one of its acute $\angle^s$, and let fall AD $\perp$ BC, one of the sides containing $\angle$ B;

Then sq. of AC shall be = sq^s of CB, BA less twice rect. CB·BD.

First. If AD falls within $\triangle$ ABC,



Then CB being div^d into two parts in D,

sq^s of CB and BD = twice rect. CB·BD tog. with sq. of DC; (II. 7)

add to each sq. of DA,

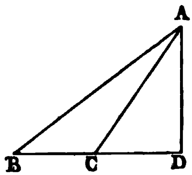
then sq^s of CB, BD and DA = twice rect. CB·BD tog. with sq^s of CD and DA:

$$\left.\begin{array}{l}\text{but sq}^s\text{ of BD and DA = sq. of BA}\\ \text{and sq}^s\text{ of CD and DA = sq. of AC}\end{array}\right\}\text{(I. 47)}$$

∴ sq^s of CB and BA = twice rect. CB·BD tog. with sq. of AC,

or sq. of AC = sq^s of CB and BA less twice rect. CB·BD... q. e. d.

Secondly. If AD fall without the △ ABC.

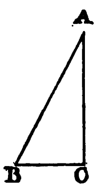


Then, BD being div^{d} into two parts in C,
sq^{s} of CB and BD = twice rect. CB·BD tog. with sq. of CD; (II. 7)
add to each sq. of DA,
then sq^{s} of CB, BD and DA = twice rect. CB·BD tog. with sq^{s} of CD and DA :

$$\left.\begin{array}{l}\text{but sq}^{s}\text{ of BD and DA} = \text{sq. of BA}\\ \text{and sq}^{s}\text{ of CD and DA} = \text{sq. of AC}\end{array}\right\}\text{(I. 47)}$$

∴ sq^{s} of CB and BA = twice rect. CB·BD tog. with sq. of AC,
or sq. of AC = sq^{s} of CB and BA less twice rect. CB·BD . . . q. e. d.

Lastly. If AC be ⊥ BC.



Then BC is *the st. line* between the perpendicular and acute ∠ B,
and sq. of AB = sq^{s} of AC and CB, (I. 47)
add to each sq. of BC,
then sq^{s} of AB and BC = sq. of AC tog. with twice sq. BC,
or sq. of AC = sq^{s} of AB and BC less twice sq. of BC.
i.e., sq. of AC = sq^{s} of AB and BC less twice rect. CB·BC.
Wherefore, in every triangle, &c.

Q. E. D.

PROP. XIV. PROBLEM.

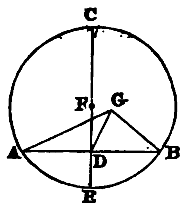
To describe a square that shall be equal to a given rectilineal figure.

BOOK III.

PROP I. PROBLEM.

To find the centre of a given circle.

Let ABC be the given ⊙:
it is req^{d} to find its centre.



Take any chord AB, and bisect it in D; (I. 10)
thro' D draw DC ⊥ AB, (I. 11)
prod. CD to meet the ⊙^{ce} in E, and bisect CE in F;
Then F shall be the centre of the ⊙.
For, if not, suppose G to be the centre, and join GA, GD, GB.
Now, in the △^{s} GDA, GDB,
∵ DA = DB (*constr.*), and DG is common to both △^{s},
we have two sides GD, DA = two sides GD, DB, ea. to ea.,
and base GA = base GB, (*radii*)
∴ ∠ GDA = GDB. (I. 8)

But when a st. line meets another st. line, making the adj^t $\angle^s$ equal to one another, each of the $\angle^s$ is a r^t $\angle$; (*def.*)

$\therefore$ $\angle$ GDB is a r^t $\angle$:

but $\angle$ FDB is also a r^t $\angle$, (*constr.*)

$\therefore$ $\angle$ FDB = GDB, (*axiom* 1)

wh. is impossible, (*axiom* 9)

$\therefore$ G cannot be the centre of $\odot$ ABC.

In the same manner it can be shown that no other p^t out of st. line CE can be the centre,

$\therefore$ the centre must be in CE;

and since CE is bisected in F,

$\therefore$ F *is* the centre of the $\odot$ ABC.

Q. E. F.

COR. Hence it is manifest, that if in a circle a straight line bisects another at right angles, the centre of the circle is in the line which bisects the other.

PROP. II. THEOREM.

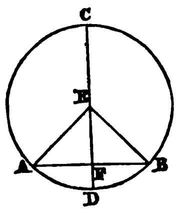
If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

PROP. III. THEOREM.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles: and, conversely, *if it cuts it at right angles, it shall bisect it.*

Let ABC be a ⊙; and let st. line CD, wh. passes thro' the centre, bisect another st. line AB, which does not pass thro' the centre in the p^t F;

Then CD shall be ⊥ AB.



Take E the centre of ⊙ (III. 1), and join EA, EB.
Now, in the △^{s} EAF, EBF,
∵ FA = FB (*hyp.*) and EF is common to both △^{s},
we have two sides EF, FA = two sides EF, FB, ea. to ea.,
and base EA = base EB, (*radii*)
∴ ∠ EFA = EFB, (I. 8)
and they are adj^t ∠^{s},
∴ CD is ⊥ AB (*def.*) . . . . q. e. d.
Conversely, if CD ⊥ AB,
AF shall be = FB.
For, the same construction being made,
∵ EA = EB, (*radii*)
∴ ∠ EAF = EBF: (I. 5)
and ∠ EFA = EFB; (*being r^t ∠^{s}*)
hence, in the two △^{s} EFA, EFB,
we have two ∠^{s} EFA, EAF = two ∠^{s} EFB, EBF, ea. to ea.,
and side EF common to both △^{s},
∴ the △^{s} are equal in all respects; (I. 26)
conseq. side AF = side FB.
Wherefore, if a straight line, &c.

Q. E. D.

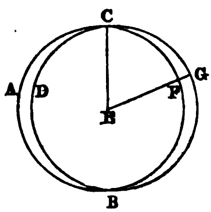
PROP. IV. THEOREM.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

PROP. V. THEOREM.

If two circles cut one another, they shall not have the same centre.

Let the two ⊙^{s} ABC, CDG cut one another in p^{ts} B and C.
They shall not have the same centre.



For, if it be possible, let any p^t E be their common centre: join EC, and draw any st. line EFG meeting the ⊙^{ces} in F and G.

Now, E being centre of ⊙ ABC,
EC = EF; (*radii*)
and E being centre of ⊙ CDG,
EC = EG; (*radii*)
but, *from above*, EC = EF,
∴ EF = EG: (*axiom* 1)
wh. is impossible. (*axiom* 9)
∴ E is not centre of ⊙^{s} ABC, CDG.

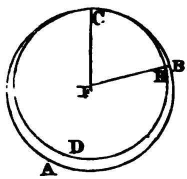
Wherefore, if two circles, &c.

Q. E. D.

PROP. VI. THEOREM.

If one circle touch another internally, they shall not have the same centre.

Let ⊙ CDE touch ⊙ ABC internally in p^t C;
They shall not have the same centre.



For, if it be possible, let any p^t F be their common centre; join FC, and draw any st. line FEB meeting the ⊙^{ces} in E and B.

Now, F being centre of ⊙ CDE,
FC = FE; (*radii*)
and F being centre of ⊙ ABC,
FC = FB; (*radii*)
but, *from above*, FC = FE,
∴ FE = FB: (*axiom* 1)
wh. is impossible. (*axiom* 9)
∴ F is not centre of ⊙^{s} ABC, CDE.

Wherefore, if two circles, &c.

Q. E. D.

PROP. VII. THEOREM.

If any point be taken in the diameter of a circle, which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least: and of any others, that which is nearer to the line which passes through the centre is always greater than one more remote.

Let AD be a diam^r of ⊙ ABCD, in wh. let any p^t F be taken, wh. is not the centre:

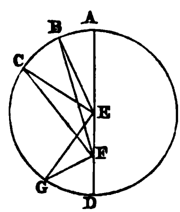
let the centre be E.

Then of all the st. lines FB, FC, FG, &c., that can be drawn to the ⊙^{ce},

FA, in wh. the centre is, shall be the greatest,

and FD, the other part of the diam^r AD, the least:

and of the others, FB, the nearer to FA, shall be > FC, the more remote, and FC > FG.



Join BE, CE, GE.

Now, BE and EF are tog. > BF; (I. 20)

but AE = BE, (*radii*)

∴ AE, and EF or AF > BF:

i.e., FA is greater than *any* st. line FB drawn from F to the ⊙^{ce}. . . . . . q. e. d.

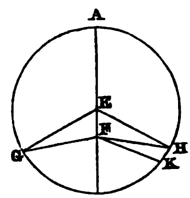
Again, in the Δ^s BEF, CEF,
$\because$ BE = CE (*radii*), and EF is common to both Δ^s,
we have two sides BE, EF = two sides CE, EF, ea. to ea.,
but $\angle$ BEF > CEF. (*axiom* 9)
$\therefore$ base BF > CF. (I. 24)
Sim^{ly}, CF > GF. . . . q. e. d.
Next, $\because$ GF and FE are tog. > GE (I. 20)
and EG = ED (*radii*)
$\therefore$ GF, FE are tog. > ED:
take away the common part FE,
$\therefore$ GF > FD: (*axiom* 5)

i.e., FD is less than *any* st. line FG that can be drawn from F to the $\odot^{ce}$ . . . . q. e. d.

Wherefore, if any point be taken, &c.

Q. E. D.

Cor. FG being any st. line drawn from F to the $\odot^{ce}$, there can be one, but only one, st. line drawn, equal to FG, on the opp. side of the diam^r.



Let E be the centre, and join EG:
at p^t E in EF make $\angle$ FEH = FEG:
Join FH.
Then, in the two Δ^s FEG, FEH,
we have two sides FE, EG = two sides FE, EH, ea. to ea.,
and $\angle$ FEG = FEH, (*constr.*)
$\therefore$ base FG = base FH. (I. 4)

But, besides FH, no other st. line can be drawn from F to the $\odot^{ce}$ = FG;

for, if possible, let it be FK,
then $\because$ FK = FG
and FG = FH
$\therefore$ FK = FH:

i.e., a st. line nearer the centre equal one that is more remote, wh. has been proved to be impossible.

PROP. VIII. THEOREM.

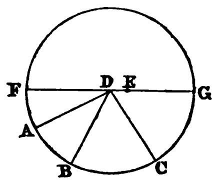
If any point be taken without a circle, and straight lines be drawn from it to the circumference whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote: but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote: and from the same point there cannot be drawn to the circumference more than two straight lines, which are equal to one another, viz., *one upon each side of the line which passes through the centre.*

PROP. IX. THEOREM.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

From p^t D within ⊙ ABC let more than two equal st. lines, *viz.*, DA, DB, DC fall to the ⊙^ce.

Then D is the centre of ⊙ ABC.



For, if not, let E be the centre;
join DE, and prod. it to ⊙^ce in F and G.
Now, FG is a diam^r of ⊙ ABC, (*def.*)
and D is a p^t in it, wh. is not the centre,
∴ DG is the greatest st. line from D to ⊙^ce,
and DC > DB, and DB > DA: (III. 7)
wh. is impossible, since they are *equal by hyp.*;
∴ E cannot be the centre.

In like manner, it may be demonstrated that no other p^t but D can be the centre,

∴ D *is* the centre of ⊙ ABC.

Wherefore, if a point be taken, &c.

Q. E. D.

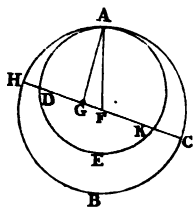
PROP. X. THEOREM.

One circumference of a circle cannot cut another in more than two points.

PROP. XI. THEOREM.

If two circles touch each other internally, the straight line which joins their centres, being produced, shall pass through the point of contact.

Let the two ⊙s ABC, ADE touch each other internally in pt A.
Then the st. line wh. joins their centres, being prodd, shall pass thro' A.



For, if not, suppose it to have another direction as HC, G being the centre of ⊙ ADE, and F the centre of ⊙ ABC.

Join AF, AG.

Now, the two sides FG, GA are tog. > AF, (I. 20)
and GA = GD, (*radii of* ⊙ ADE)
∴ FG, GD are tog. > AF,
i.e., FD > AF;
but, again, AF = FH, (*radii of* ⊙ ABC)
∴ FD > FH;
wh. is impossible.

∴ the st. line wh. joins the centres of the ⊙s cannot have the direction HC; and, in the same manner, it may be shown that it cannot have any other direction wh. does not pass thro' A.

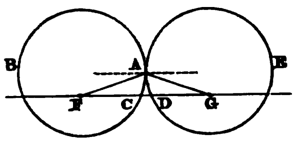
Wherefore, if two circles touch one another internally, &c.

Q. E. D.

PROP. XII. THEOREM.

If two circles touch each other externally, the straight line which joins their centres, shall pass through the point of contact.

Let the two circles ABC, ADE touch each other externally in p^t A.
Then the st. line, wh. joins their centres, shall pass thro' A.



For, if not, suppose it to have another direction as FG.
F being the centre of ⊙ ABC, and G the centre of ⊙ ADE.

Join AF, AG.

Now, AF, AG are tog. > FG, (I. 20)
but AF = FC, (*radii of* ⊙ ABC)
and AG = GD, (*radii of* ⊙ ADE)
∴ FC, GD are tog. > the whole line FG,
wh. is impossible.

∴ the st. line, wh. joins the centres of the ⊙^{s}, cannot have the direction FG; and, in the same manner, it may be shown that it cannot have any other direction wh. does not pass thro' A.

Wherefore, if two circles touch each other externally, &c.

Q. E. D.

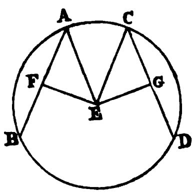
PROP. XIII. THEOREM.

Two circles cannot touch each other, internally or externally, in more than one point.

PROP. XIV. THEOREM.

Equal straight lines in a circle, or equal chords, *are equally distant from the centre, and,* conversely, *chords which are equally distant from the centre are equal to one another.*

Let AB, CD be equal chords in ⊙ ABDC,
take E the centre of the ⊙, draw EF ⊥ AB, and EG ⊥ CD:
Then the chords AB, CD shall be equally distant from the centre;
that is, EF shall be = EG.



Join AE, CE.

Now, EF being ⊥ AB, AF is half AB } (III. 3)
and EG being ⊥ CD, CG is half CD }
but the halves of equal things are themselves equal.
∴ AF = CG,
∴ sq. of AF = sq. of CG.
Again, ∵ AE = CE, (*radii*)
∴ sq. of AE = sq. of CE;
but sq. of AE = sq^s of AF and FE, } I. 47
and sq. of CE = sq^s of CG and GE, }
∴ sq^s of AF and FE are tog. = sq^s of CG and GE;
but, *from above*, sq. of AF = sq. of CG,
∴ rem^g sq. of FE = rem^g sq. of GE; (*axiom* 3)
∴ FE = GE.

But st. lines are equally dist. from the centre of a ⊙ when the ⊥^{s} drawn to them from the centre are equal,
∴ chords AB, CD are equally dist. from the centre.
Wherefore, equal straight lines in a circle, &c.

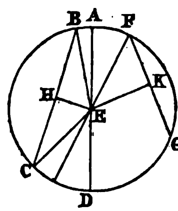
Q. E. D.

PROP. XV. THEOREM.

The diameter is the greatest straight line in a circle; and of all chords, that which is nearer to the centre is always greater than one more remote; and, conversely, *the greater chord is nearer to the centre than the less.*

Let ABCG be a ⊙, E its centre, AD its diam^r, BC, FG chords, of wh. BC is nearer to the centre than FG.

Then AD shall be > *any* chord BC, and BC shall be > FG.



Draw EH ⊥ BC, and EK ⊥ FG, (I. 12)

join EB, EF, EC.

$$\left.\begin{array}{r}\text{Now, EB} = \text{EA}\\ \text{and EC} = \text{ED}\end{array}\right\}\textit{(radii)}$$

∴ EB, EC are tog. = diam^s AD : (*axiom* 2)

but EB, EC are tog. > BC, (I. 20)

∴ diam^r AD > any chord BC . . . q. e. d.

Again, BC being nearer to the centre than FG, (*hyp.*)

EH < EK, (*def.*)

∴ sq. of EH < sq. of EK;

and EB = EF, (*radii*)

∴ sq. of EB = sq. of EF:

$$\left.\begin{array}{l}\text{but sq. of EB} = \text{sq}^{\text{s}} \text{ of EH and HB,}\\ \text{and sq. of EF} = \text{sq}^{\text{s}} \text{ of EK and KF,}\end{array}\right\}\text{(I. 47)}$$

∴ sqˢ of EH and HB are tog. = sqˢ of EK and KF:

But, *from above*, sq. of EH < sq. of EK,

∴ sq. of HB *must be* > sq. of KF,

and ∴ HB > KF:

$$\left.\begin{array}{l}\text{but BC is double of HB,}\\ \text{and FG is double of FK,}\end{array}\right\}\text{(III. 3)}$$

and the doubles of unequal things are, *à fortiori*, unequal,

∴ BC > FG. . . . q. e. d.

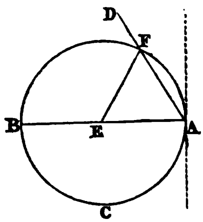
Wherefore, the diameter is the greatest straight line in a circle, &c.

Q. E. D.

PROP. XVI. THEOREM.

A straight line drawn at right angles to the diameter of a circle, at its extremity, falls outside the circle; and no straight line can be drawn between it and the circumference so as not to cut the circle.

Let ABC be a ⊙, E its centre, and AB its diam^r.
Then the st. line, drawn ⊥ AB from its ext^y A, shall fall outside the circle.



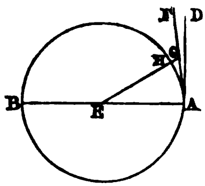
For, if not, it must either coincide with the ⊙^{ce}, or fall within (*i.e.*, *cut*) the ⊙.

Now, as a st. line cannot, *for however small a space*, coincide with a curved line, a st. line, drawn ⊥ AB from A, cannot coincide with the ⊙^{ce}.
Suppose it, then, to fall within the ⊙, as AD, cutting the ⊙^{ce} in F.

Join EF.
Now, ∵ EA = EF, (*radii*)
∴ ∠ EAF = EFA; (I. 5)
But EAF is a r^t ∠, (*hyp.*)
∴ EFA is a r^t ∠;
conseq. two ∠^{s} of △ EAF are tog. = two r^t ∠^{s};
wh. is impossible. (I. 17)

∴ AD cannot fall within the ⊙, and, as it cannot coincide with the ⊙^{ce}, it must fall outside the ⊙, *as the dotted line in the fig.* . . . q. e. d.

Also, no straight line can be drawn between it and the ⊙ce from the pt A, so as not to cut the ⊙.
For, suppose AF to be drawn between AD and the ⊙ce, meeting the ⊙ce in A, but not cutting the ⊙.



Draw EG ⊥ AF.

Now, since every pt in AF is outside the ⊙ (*hyp.*)
Some portion, HG *for instance*, of the ⊥r EG, must extend beyond the ⊙ce.
Hence EGA being a rt ∠ (*constr.*) and EAG < a rt ∠, (I. 17)

∴ side EA > EG, (I. 19)
but EA = EH, (*radii*)
∴ EH > EG,
wh. is impossible. (*axiom* 9)

∴ no st. line can be drawn from the pt A, between AD and the ⊙ce, which does not cut the ⊙ ABC.

Wherefore, a straight line drawn at right angles, &c.

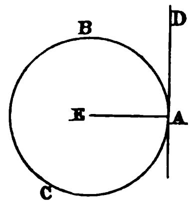
Q. E. D.

Cor. Hence, it is manifest that a st. line, which is perpendicular to the diameter or to the radius of a circle, at its extremity touches the circle (*def.*); and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it (III. 2): also that there can be but one straight line touching a circle at the same point.

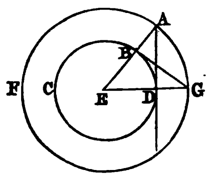
PROP. XVII. PROBLEM.

To draw a straight line from a given point, either in the circumference or outside of it, which shall touch a given circle.

First, let the given p^{t} A be in the $\odot^{ce}$ of $\odot$ ABC:
it is req^{d} to draw a st. line from A, wh. shall touch the $\odot$.



Take E the centre (III. 1), join EA,
and at p^{t} A draw AD $\perp$ EA:
AD touches the $\odot$ (III. 16. Cor.)... q. e. f.
Secondly, let the given p^{t} A be outside the $\odot^{ce}$ of $\odot$ BCD.



Take E the centre (III. 1); join EA;
with centre E, and dist. EA, desc. $\odot$ AFG;
from B draw BG $\perp$ AE; join EDG, AD:
Then AD *touches* the $\odot$ BCD.
For, in the two Δ^{s} EAD, EGB, we have
two sides DE, EA = two sides BE, EG, ea. to ea.,
and $\angle$ E common to both Δ^{s},
$\therefore$ the Δ^{s} are equal in all respects; (I. 4)
conseq. $\angle$ EDA = EBG,
but EBG is a r^{t} $\angle$, (*constr.*)
$\therefore$ EDA is a r^{t} $\angle$.

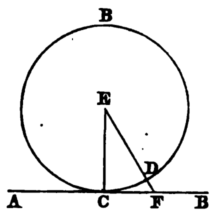
And st. line DA, being, therefore, $\perp$ to rad. ED at its ext^{y}, *touches* the $\odot$ BCD, (III. 16, Cor.) and it is drawn from the given p^{t} A.

Q. E. F.

PROP. XVIII. THEOREM.

If a straight line touch a circle and another straight line be drawn from the centre to the point of contact, the two straight lines shall be at right angles to each other.

Let st. line AB touch ⊙ BCD in p^{t} C;
take E the centre, and join EC.
Then EC shall be $\perp$ AB.



For, if not, draw EF $\perp$ AB.
Now, EFC being a r^{t} $\angle$, (*constr.*)
ECF < a r^{t} $\angle$, (I. 17)
$\therefore$ EC > EF, (I. 19)
but EC = ED, (*radii*)
$\therefore$ ED > EF,
wh. is impossible. (*axiom* 9)

$\therefore$ EF cannot be $\perp$ AB, and, in the same manner, it may be shown that no other st. line but EC can be $\perp$ AB;

conseq. EC *is* $\perp$ AB,
or EC and AB are at r^{t} $\angle^{s}$ to each other.

Wherefore, if a straight line touch a circle, &c.

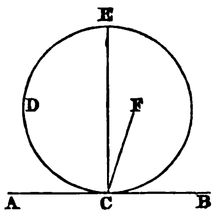
Q. E. D.

PROP. XIX. THEOREM.

If a straight line touch a circle and another straight line be drawn from the point of contact at right angles to it, the centre of the circle shall be in this line.

Let st. line AB touch ⊙ CDE in p^t C, and let CE be drawn ⊥ AB:

Then the centre of the ⊙ shall be in CE.



For, if not, suppose any p^t F, wh. is not in CE, to be the centre, and join CF.

Now, FCB is a r^t ∠, (III. 18)

but ECB is a r^t ∠, (*hyp.*)

∴ ∠ FCB = ECB,

wh. is impossible. (*axiom* 9)

∴ F cannot be the centre, and, in the same manner, it may be shown that no other p^t, except it be in CE, can be the centre,

∴ the centre *is* in CE.

Wherefore, if a straight line touch a circle, &c.

Q. E. D.

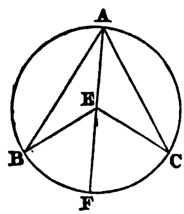
PROP. XX. THEOREM.

The angle at the centre of a circle is double of the angle at the circumference, the angles standing on the same arc.

Let ABC be a ⊙, and let the angles, BEC at its centre and BAC at its ⊙ce, stand on the same arc BC.

Then the ∠ BEC shall be double of the ∠ BAC.

First, let the centre E fall within the ∠ BAC.



Join AE, and prod. it to F.

Now, ∵ EA = EB, (*radii*)

∴ ∠ EAB = EBA; (I. 5)

∴ ∠s EAB, EBA are tog. double of EAB;

but ∠ BEF = ∠s EAB, EBA, (I. 32)

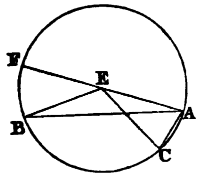
∴ ∠ BEF is double of ∠ BAE;

Simly it may be shown, that

∠ CEF is double of ∠ CAE;

evidently, ∴ the whole ∠ BEC is double of ∠ BAC. (q. e. d.)

Secondly, let the centre E fall outside the ∠ BAC.



Join AE, and prod. it to F.

Now, it may be shown, *as before*, that

∠ FEC is double of ∠ FAC,

and that ∠ FEB is double of ∠ FAB,

from whole ∠ FEC take ∠ FEB, and ∠ BEC remains;

from whole ∠ FAC take ∠ FAB, and ∠ BAC remains;

evidently, ∴ ∠ BEC is double of ∠ BAC. (q. e. d.)

Wherefore, the angle at the centre of a circle, &c.

Q. E. D.

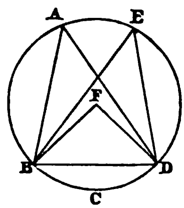
PROP. XXI. THEOREM.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a ⊙, and BAD, BED ∠s in same seg^t BAED;

Then ∠ BAD shall be = BED.

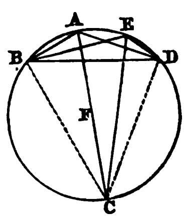
First. Let seg^t BAED be > semicircle.



Take F the centre of ⊙ (III. 1), and join BF, DF.
Now, ∠ BFD at the centre and ∠ BAD at the ⊙^{ce} stand on same arc BCD,

∴ ∠ BFD is double of ∠ BAD; (III. 20)
Sim^{ly}, ∠ BFD is double of ∠ BED:
∴ ∠ BAD = BED. (*axiom* 7)

Secondly. Let seg^t BAED be not > semicircle.



Join AF; prod. it to C, and join CE.
Now seg^t BADC is > semicircle,
∴ ∠ BAC = BEC, *from above;*
Sim^{ly}, seg^t CBED is > semicircle,
∴ ∠ CAD = CED:
∴ whole ∠ BAD = whole ∠ BED. (*axiom* 2)

Wherefore, angles in the same segment, &c.

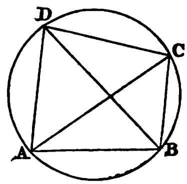
Q. E. D.

PROP. XXII. THEOREM.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quad[l] fig. insc[d] in ⊙ ABCD.

Then any two of its opp. ∠[s] shall be tog. = two r[t] ∠[s].



Join AC, BD.

$$\left.\begin{array}{r}\text{Now, } \angle\ \text{ACB} = \text{ADB,} \\ \text{and } \angle\ \text{BAC} = \text{BDC,}\end{array}\right\}\ \text{(III. 21)}$$

hence the two ∠[s] BAC, ACB are tog. = whole ∠ ADC;

add to each ∠ CBA,

then the three ∠[s] BAC, ACB, CBA are tog. = the two ∠[s] ADC, CBA;

but the three ∠[s] BAC, ACB, CBA are tog. = two r[t] ∠[s], (I. 32)

∴ the ∠[s] ADC, CBA are tog. = two r[t] ∠[s]. (*axiom* 1)

In the same manner it may be shown that

the ∠[s] BAD, BCD are tog. equal to two r[t] ∠[s].

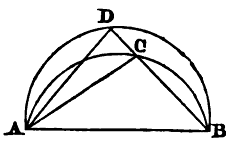
Wherefore, the opposite angles, &c.

Q. E. D.

PROP. XXIII. THEOREM.

Upon the same chord, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, upon the same chord AB, and upon the same side of it, let there be two similar seg^{ts} of $\odot^s$ ACB, ADB, not coinciding with one another.



Then, as the $\odot^s$ of wh. ACB, ADB are seg^{ts} cut one another in two p^{ts} A, B, they cannot cut one another in any other p^t; (III. 10)

hence one seg^t *must* fall within the other:
let ACB fall within ADB;
draw any st. line BCD, and join AC, AD.
Now, $\because$ seg^t ACB is similar to seg^t ADB, (*hyp.*)
and similar seg^{ts} of $\odot^s$ contain equal $\angle^s$, (*def.*)
$\therefore$ $\angle$ ACB = ADB,
the ext^r $\angle$ equal to the int^r opp. $\angle$,
wh. is impossible. (I. 16)

Wherefore, there cannot be two similar seg^{ts} of $\odot^s$ upon the same side of the same chord, which do not coincide.

Q. E. D.

PROP. XXIV. THEOREM.

Similar segments of circles upon equal chords are equal to one another.

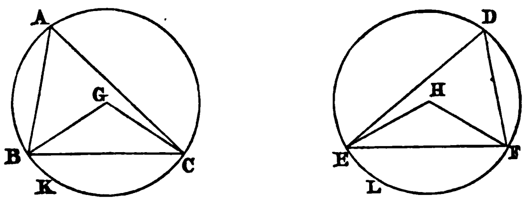
PROP. XXV. PROBLEM.

A segment of a circle being given, to describe the circle of which it is the segment.

PROP. XXVI. THEOREM.

In equal circles equal angles *stand upon equal arcs, whether they be at the centres or circumferences.*

Let ⊙ ABC = ⊙ DEF,
and let BGC, EHF be equal ∠s at their centres,
and BAC, EDF equal ∠s at their ⊙ces.
Then arc BKC shall be = arc ELF.



Join BC, EF.
Now, in the △s BGC, EHF
we have two sides BG, GC = two sides EH, HF, ea. to ea.,
and ∠ G = ∠ H, (*hyp.*)
∴ base BC = base EF. (I. 4)
And ∠ A being = ∠ D, (*hyp.*)
BAC, EDF are similar segts, (*def.*)
and they are upon equal chords, BC, EF;
but similar segts of ⊙s upon equal chords are equal to one another, (III. 24)
∴ segt BAC = segt EDF:
but the whole ⊙ ABC = whole ⊙ DEF, (*hyp.*)
∴ remg segt BKC = remg segt ELF, (*axiom* 3)
∴ arc BKC = arc ELF.
Wherefore, in equal circles, &c.

Q. E. D.

PROP. XXVII. THEOREM.

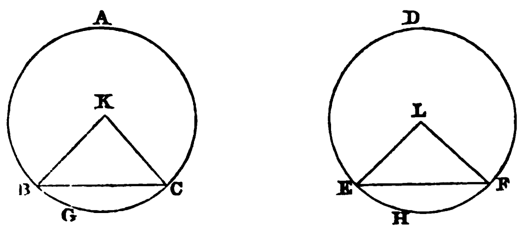
In equal circles, the angles which stand upon equal arcs are equal to one another, whether they be at the centres or circumferences.

PROP. XXVIII. THEOREM.

In equal circles, equal chords *cut off equal arcs, the greater equal to the greater, and the less to the less.*

Let ⊙ ABC = ⊙ EDF,
and let BC, EF be equal chords, wh. cut off the two greater arcs BAC, EDF, and the two less BGC, EHF.

Then the greater arc BAC shall be = the greater EDF,
and the less BGC shall be = the less EHF.



Take K, L the centres (III. 1), and join BK, KC, EL, LF.
Now, in the △s BKC, ELF,
we have two sides BK, KC = two sides EL, LF, ea. to. ea.,
and base BC = base EF, (*hyp.*)
∴ ∠ K = ∠ L: (I. 8)
but, in equal ⊙s, equal ∠s stand upon equal arcs, whether they be at the centres or at the ⊙ces, (III. 26)
∴ less arc BGC = less arc EHF; . . . . q. e. d.
but whole ⊙ce ABC = whole ⊙ce DEF, (*hyp.*)
∴ remg arc BAC = remg arc EDF. (*axiom* 3)
Wherefore, in equal circles, &c.

Q. E. D.

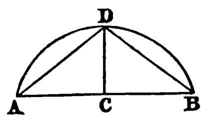
PROP. XXIX. THEOREM.

In equal circles, equal arcs are subtended by equal chords.

PROP. XXX. PROBLEM.

To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be a given arc.
It is req^{d} to bisect it.



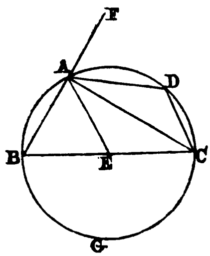
Join AB, and bisect it in C; (I. 10)
from C draw CD $\perp$ AB; (I. 11)
Join AD, DB.
Now, in the $\triangle^{s}$ ACD, BCD,
$\because$ AC = CB (*constr.*), and CD is common to both $\triangle^{s}$,
we have two sides AC, CD = two sides BC, CD, ea. to ea.,
and $\angle$ ACD = BCD, each of them being a r^{t} $\angle$, (*constr.*)
$\therefore$ base AD = base BD : (I. 4)
but in equal $\odot^{s}$, equal chords cut off equal arcs, the greater equal to greater, and less to less, (III. 28)
and AD, DB are, *each of them*, < a semicircle,
since DC passes thro' the centre, (III. 1. Cor.)
$\therefore$ arc AD = arc DB.
Therefore, the given arc ADB is bisected in D.

Q. E. F.

PROP. XXXI. THEOREM.

The angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is an acute *angle: and the angle in a segment less than a semicircle is an* obtuse *angle.*

Let BGC be a ⊙, E its centre, BC its diam^r:
from B draw any st. line BF cutting the ⊙^{ce} in A;
in arc AC take any p^t D, and join AE, AC, AD, DC.
Then ∠ BAC, *in a semicircle*, is a r^t ∠ ;
∠ B, *in a segm^t* > *a semi*⊙, is an acute ∠ ;
∠ D, *in a segm^t* < *a semi*⊙, is an obtuse ∠.



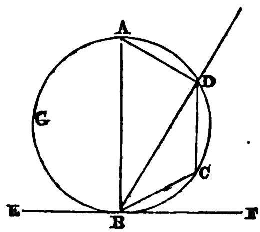
For, ∵ EB = EA, (*radii*)
∴ ∠ EBA or CBA = EAB (I. 5)
Sim^{ly}, ∠ BCA = EAC,
∴ whole ∠ BAC = the two ∠^{s} ABC, ACB; (*axiom* 2)
but ∠ FAC = the same two ∠^{s}, (I. 32)
∴ ∠ BAC = FAC,
and they are adj^t angles,
∴ BAC is a r^t ∠. (*def.*) . . . . q. e. d.
Conseq. ∠ B, of △ ABC, *must* be an acute ∠, (I. 17). . . q. e. d.
and ABCD, being a quad^l fig. insc^d in a ⊙,
its opp. ∠^{s}, B and D, are tog. = two r^t ∠^{s}, (III. 22)
and, since, *from above*, ∠ B < a r^t ∠,
∴ ∠ D *must* be > a r^t ∠, (*i.e.* an obtuse ∠) . . . . q. e. d.
Wherefore, the angle in a semicircle is a right angle, &c.
Q. E. D.

COR. From this it is manifest, that if one angle of a triangle be equal to the other two it is a right angle: because the angle adjacent to it is equal to the same two; (I. 32)—and when the adjacent angles are equal they are right angles. (*def.*)

PROP. XXXII. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

Let st. line EF touch the ⊙ ADB in p^t B;
and from B draw a st. line BD cutting the ⊙ :
At B draw BA ⊥ EF, in arc BD take any p^t C, and join AD, DC, CB.
Then ∠ DBF shall be = ∠ A, wh. is in the alt. segm^t DGB,
and ∠ DBE shall be = ∠ C, wh. is in the alt. segm^t DCB.



For, ∵ EF touches the ⊙, and BA is ⊥ EF,
∴ BA is a diam^r of the ⊙ ; (III. 19)
and ADB, an ∠ in a semi⊙, is a r^t ∠ ; (III. 31)
conseq. the two rem^g ∠^{s} BAD, DBA, of △ ABD, are tog. = a r^t ∠ ; (I. 32)
but ABF is a r^t ∠, (*constr.*)
∴ ∠ ABF = the two ∠^{s} BAD, DBA; (*axiom* 1)
take from each the com. ∠ DBA,
then, rem^g ∠ DBF = rem^g ∠ BAD: (*axiom* 3) . . . . q. e. d.
and ABCD being a quad^l fig. insc^d in a ⊙,
its two opp. ∠^{s} BAD, DCB are tog. = two r^t ∠^{s}; (III. 22)
but ∠^{s} DBF, DBE are tog. = two r^t ∠^{s}, (I. 13)
∴ ∠^{s} DBF, DBE are tog. = the ∠^{s} BAD, DCB;
but, *from above*, ∠ DBF = BAD,
∴ rem^g ∠ DBE = rem^g ∠ DCB. . . . . q. e. d.
Wherefore, if a straight line touch a circle, &c.

Q. E. D.

PROP XXXIII. PROBLEM.

Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

PROP. XXXIV. PROBLEM.

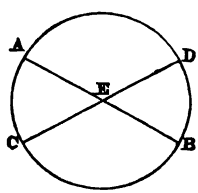
From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.

PROP. XXXV.

If two straight lines cut one another within a circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

Let st. lines AB, CD cut one another in p^t E within the ⊙ ACBD.

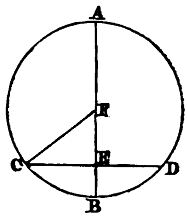
Then the rect. AE·EB shall = rect. CE·ED.



First, if both the st. lines pass thro' the centre, it is *manifest* that the rect^s cont^d by their parts are sq^s of radii,

∴ rect. AE·EB = rect. CE·ED. . . . (q. e. d.)

Secondly, let one of the st. lines AB, wh. passes thro' the centre F, be $\perp$ to the other st. line CD, wh. does not pass thro' the centre.



Join CF.

Now, $\because$ AB is div[d] into equal parts in F,

and into unequal parts in E,

$\therefore$ rect. AE·EB tog. with sq. of FE = sq. of rad. FB, (II. 5)

= sq. of rad. FC,

= sq[s] of CE and FE; (I. 47)

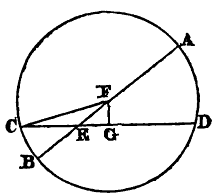
take from each the com. sq. of FE,

then, rect. AE·EB = sq. of CE;

but sq. of CE = rect. CE·ED, (III. 3)

$\therefore$ rect. AE·EB = rect. CE·ED. . . . . q. e. d.

Thirdly, when AB, wh. passes thro' the centre F, is not $\perp$ CD, wh. does not pass thro' the centre.



Draw FG $\perp$ CD (I. 12), and join CF.

Now, CD is bisected in G; (III. 3)

∴ *as above*, rect. CE·ED tog. with sq. of EG = sq. of CG; (II. 5)

add to each the sq. of GF,

then, rect. CE·ED, tog. with sq^s of EG and GF = sq^s of CG and GF;

$$\left.\begin{array}{l}\text{but sq}^s\text{ of EG and GF} = \text{sq. of EF,}\\ \text{and sq}^s\text{ of CG and GF} = \text{sq. of CF,}\end{array}\right\}\text{(I. 47)}$$

∴ rect. CE·ED tog. with sq. of EF = sq. of rad. CF,

= sq. of rad. BF;

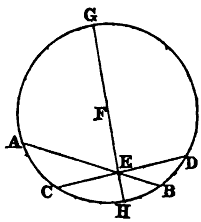
but, rect. AE·EB tog. with sq. of EF = sq. of BF, (II. 5)

∴ rect. AE·EB tog. with sq. of EF = rect. CE·ED tog. with sq. of EF;

take from each the com. sq. of EF,

then, rect. AE·EB = rect. CE·ED. . . . . (q. e. d.)

Lastly, let neither of the st. lines pass thro' the centre F.



Join FE, and prod. it both ways to the ⊙^{ce}.

$$\left.\begin{array}{l}\text{Now, rect. AE·EB} = \text{rect. GE·EH}\\ \text{and rect. CE·ED} = \text{rect. GE·EH}\end{array}\right\}\textit{from above}$$

∴ rect. AE·EB = rect. CE·ED . . . . . (q. e. d.)

Wherefore, if two st. lines cut one another within a circle, &c.

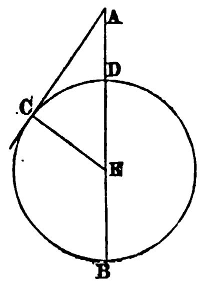
Q. E. D.

PROP. XXXVI. THEOREM.

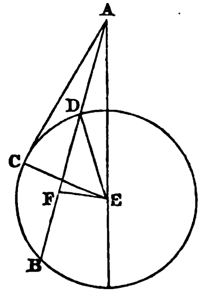
If from a point, outside a circle, two straight lines be drawn, one of which cuts and the other touches the circle; the rectangle, contained by the whole cutting line and the part outside the circle, shall be equal to the square of the line which touches the circle.

From p^t A, outside the ⊙ BCD, let the st. lines AB, AC be drawn, AB cutting the ⊙ and AC touching it.

Then the rect. BA·AD shall be = sq. of AC.



First, let AB pass thro' the centre E, and join CE,
then ACE is a r^t ∠ : (III. 18)
and ∵ st. line BD is bisected in E and prod^d to A,
∴ rect. BA·AD tog. with sq. of ED = sq. of AE. (II. 6)
But sq. of ED = sq. of EC, (*radii*)
and sq. of EA = sq^s of AC and EC, (I. 47)
∴ rect. BA·AD tog. with sq. of EC = sq^s of AC and EC; (*axiom* 1)
take from each the com. sq. of EC,
then, rect. BA·AD = sq. of AC. (*axiom* 3) . . . (q. e. d.)

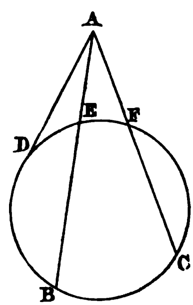


Secondly, where AB does not pass thro' the centre E;
draw EF ⊥ AB, (I. 12) and join EA, ED, EC.
Now, BD being bisected in F, (III. 3) and prod[d] to A,
rect. BA·AD tog. with sq. of DF = sq. of AF; (II. 6)
add to each the sq. of FE,
then, rect. BA·AD tog. with sq[s] of DF and FE = sq[s] of AF and FE; (*axiom* 2)
but, sq[s] of DF and FE are tog. = sq. of DE, } (I. 47)
and sq[s] of AF and FE are tog. = sq. of AE; }
∴ rect. BA·AD tog. with sq. of DE = sq. of AE;
but, again, sq. of DE = sq. of CE; (*radii*)
and sq. of AE = sq[s] of AC and CE, (I. 47)
∴ rect. BA·AD tog. with sq. of CE = sq[s] of AC and CE,
take from each the com. sq. of CE,
then, rect. BA·AD = sq. of AC.... (q. e. d.)
Wherefore, if from a point outside a circle, &c.

Q. E. D.

COR. If from a point outside a circle there be drawn two st. lines cutting it, as AB, AC,

rect. BA·AE = rect. CA·AF.



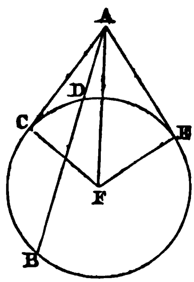
For, if AD touch the ⊙,
each of the rectangles = sq. of AD.

PROP. XXXVII. THEOREM.

If from a point outside a circle there be drawn two straight lines, one of which cuts the circle and the other meets *it, and the rectangle, contained by the whole cutting line and the part outside the circle, be equal to the square of the line which meets the circle; the line which meets shall touch the circle.*

From p^t^ A, outside the ⊙ BCD, let st. lines AB, AC be drawn, AB cutting the ⊙, AC *meeting* it, and let rect. BA·AD = sq. of AC.

Then AC shall touch the ⊙.



Take the centre F, draw AE touching the ⊙, (III. 17)
and join FC, FA, FE.
Now, AEF is a r^t^ ∠, (III. 18)

and ∵ { rect. BA·AD = sq. of AC, (*hyp.*)
rect. BA·AD = sq. of AE, (III. 36) }

∴ sq. of AC = sq. of AE,
∴ AC = AE,
and CF = EF. (*radii*)
Hence, in Δ^s^ ACF, AEF, we have
two sides AC, CF = AE, EF, ea. to ea.,
and base AF com. to both Δ^s^,
∴ ∠ ACF = AEF, (I. 8)
but AEF is a r^t^ ∠, (*constr.* (III. 18)
∴ ACF is a r^t^ ∠,
∴ AC touches the ⊙. (III. 16, Cor.)

Wherefore, if from a point outside a circle, &c.

Q. E. D.

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